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Certain Poincaré Series and Period-integrals of Eisenstein Series on $\mathbb{R}$-rank One Classical Groups

by

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1. Introduction and basic notations

1.1. Introduction

Let G be the automorphism group of a Hermitian form on a finite dimensional vector space over a division $\mathbb{Q}$-algebra with an involution and H the stabilizer of an anisotropic vector. Let us consider the integral

$$\mathcal{P}_{\mathsf{H}}(E(\nu)) = \int_{\mathsf{H}(\mathbb{Q})\backslash\mathsf{H}(\mathbb{A})} E(\nu\,;h)\,d\dot{h} \tag{1.1}$$

for the spherical Eisenstein series $E(\nu)$ ($\nu \in \mathbb{C}$) on $\mathsf{G}(\mathbb{A})$, which is a kind of the Rankin-Selberg integrals studied by Murase and Sugano among others ([14]). The integral (1.1) occurs in the spectral expansion and the functional equation of the Green function on the arithmetic quotients of complex hyperball ([16]). In this paper, under the assumption that the $\mathbb{R}$-rank of G is one, we compute the integral (1.1) to obtain its explicit evaluation. For that purpose, we adopt a different approach from [14], since the convergence-region of (1.1) is disjoint from that of $E(\nu)$. Let us explain our method giving a brief summary of each section. After two preliminary sections where we introduce basic objects including the Eisenstein series, in section 4, we study basic properties of certain Poincaré series, which is a variant of the 'hyperbolic Eisenstein series' introduced by Kudla-Milson in [13]. In section 5 and 6, we compute its constant term and study an average of the truncated Mellin transform of the constant term. In section 7, we prove a formula which relates the integral $\mathcal{P}_{\mathsf{H}}(E(\nu))$ to the iner product of the hyperbolic Eisenstein series and the usual one $E(\nu)$. In the last part of this paper, we see that the inner product is nothing but the average of the Mellin transforms of certain functions stemming from the constant terms computed in setion 6 (Theorem 2). In section 8 and 9 we give an explicit expression of the integral (1.1) in terms of some elementary zeta-functions (Theorem 3).

1.2. Basic natotions

Put $\mathbb{N} := \{n \in \mathbb{Z} | \, n \geqslant 0\}$ and $\mathbb{N}^* := \mathbb{N} - \{0\}$. For a ring R, $\mathrm{M}_{lm}(R)$ denotes the set of $l \times m$-matrices with coefficients in R. We write R^l instead of $\mathrm{M}_{l,1}(R)$. The adele ring (resp. the set of finite adeles) of $\mathbb{Q}$ is denoted by $\mathbb{A}$ (resp. $\mathbb{A}_{\mathrm{f}}$). The modulus of the p-adic field $\mathbb{Q}_p$ (resp. $\mathbb{R}$, $\mathbb{A}$, $\mathbb{A}_{\mathrm{f}}$) is denoted by $|\cdot|_p$ (resp. $|\cdot|_\infty$, $|\cdot|_{\mathbb{A}}$, $|\cdot|_{\mathrm{f}}$). For a measure space

X, the measure of a measureable subset $S \subset X$ is denoted by $\mathrm{mes}(S)$. For any algebraic $\mathbb{Q}$-group G, we put the counting measure on the discrete group $\mathsf{G}(\mathbb{Q})$. For subset A of a given set X, $\mathrm{Char}_A : X \to \{0, 1\}$ is the characteristic function of A.

2. Preliminary

2.1. Algebras

Let E and ι be one of the following.

(i) $E = \mathbb{Q}$ and $\iota = \mathrm{id}$.

(ii) E is an imaginary quadratic extention of $\mathbb{Q}$ and ι is the non trivial automorphism of E.

(iii) E is a definite quaternion division algebra over $\mathbb{Q}$ and ι is the main involution of E.

Put $\delta := \dim_{\mathbb{Q}}(E)$. We write a^ι in place of $\iota(a)$. For $a \in E$, put $\tau(a) := a + a^\iota$, $\mathrm{N}(a) := aa^\iota$. For a $\mathbb{Q}$-algebra R, set $E_R := E \otimes_{\mathbb{Q}} R$, $E_R^{(0)} := \{a \in E_R |\, \tau(a) = 0\}$ and $E_R^{(1)} := \{a \in E_R |\, \mathrm{N}(a) = 1\}$. By assumptions, $E_{\mathbb{R}} \cong \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the Hamilton quaternion algebra) according to the case (i), (ii) or (iii). Let $\mathscr{O}$ be a maximal order of E; then $\mathscr{O}^\iota = \mathscr{O}$. Let p be a prime. If $E_p := E_{\mathbb{Q}_p}$ is not division, p is said to be a split prime for E; the totality of split primes is denoted by $S(E)$. For $p \notin S(E)$, $\mathscr{O}_p := \mathscr{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order of the division algebra E_p and has a prime element ϖ. We say that p is a ramified prime or an innert prime according to $|\mathrm{N}(\varpi)|_p = p^{-1}$ or $|\mathrm{N}(\varpi)|_p = p^{-2}$. We denote $R(E)$ (resp. $I(E)$) the set of ramified primes (resp. innert primes) of E. Note that $I(E) = \varnothing$ in case (iii) and $I(\mathbb{Q})$ is the set of all prime numbers by definition. Let $D_{\mathscr{O}}$ be the discriminant of $\mathscr{O}$, that is the group index of $\mathscr{O}$ in $\mathfrak{d}_E^{-1} = \{a \in E |\, \tau(a\mathscr{O}) \subset \mathbb{Z}\}$. Note $D_{\mathbb{Z}} = 2$. Recall that if $\delta > 1$, then $R(E)$ coincides with the set of prime divisors of $D_{\mathscr{O}}$.

2.2. Hermitian forms

For a matrix $X = (x_{ij}) \in \mathrm{M}_{mn}(E)$, set $X^* := {}^t(X^\iota) = (x_{ji}^\iota) \in \mathrm{M}_{nm}(E)$. Let $S_0 \in \mathrm{M}_{qq}(E)$ be a Hermitian matrix, i.e., $S_0^* = S_0$. The bi-additive form $S_0(X, Y) := Y^* S_0 X$ on E^q satisfies $S_0(X, Y)^\iota = S_0(Y, X)$ and $S_0(X\alpha, Y\beta) = \beta^\iota S_0(X, Y)\alpha$ for $X, Y \in E^q$ and $\alpha,\ \beta \in E$. We make the three assumptions on S_0, which we keep through out this paper:

(a) S_0 is positive definite, that is $S_0[X] := S_0(X, X) > 0$ $(\forall X \in E^q - \{0\})$,

(b) S_0 is even integral, that is $S_0(X, Y) \in \mathscr{O}$, $S_0[X] \in \tau(\mathscr{O})$ $(\forall X,\ Y \in \mathscr{O}^q)$.

(c) the $\mathscr{O}$-lattice $\mathscr{O}^q$ is maximal among those $\mathscr{O}$-lattices satisfying (b).

Since $\mathscr{O}$ is a free $\mathbb{Z}$-module of rank δ, the function $Q_0(X) := 2S_0[X]$ on the $\mathbb{Z}$-module $\mathscr{O}^q$ is a quadratic form of δq variables, whose associated $\mathbb{Q}$-bi-linear form $Q_0(X, Y) = \tau(S_0(X, Y))$ is positive definite and integral. We have $\det(Q_0) = D_{\mathscr{O}}^q\, \#(S_0^{-1}\mathscr{O}^q/\mathscr{O}^q)$. By [14, Lemma 3.2 (iii) (p. 37)], (c) implies $|\det(Q_0)|_p = 1$ $(\forall p \in S(E))$.

The Hermitian matrix $S := \begin{bmatrix} & & 1 \\ & S_0 & \\ 1 & & \end{bmatrix}$ defines a bi-additive form $S(X, Y)$ on the right E-vector space $E^{q+2} := \begin{bmatrix} E \\ E^q \\ E \end{bmatrix}$ in the same way as S_0 does on E^q. It satisfies the property (b) but with $\mathscr{O}^q$ replaced by $\mathscr{L} := \begin{bmatrix} \mathscr{O} \\ \mathscr{O}^q \\ \mathscr{O} \end{bmatrix}$.

2.3. Unitary group and its subgroups

Let G be the unitary group for S, that is the $\mathbb{Q}$-algebraic group such that $\mathsf{G}(R) = \{g \in \mathrm{GL}_{q+2}(E_R) |\ g^* S g = S\}$ for any $\mathbb{Q}$-algebra R. Similarly we have the unitary group for S_0 denoted by G_0.

Given $(t, g_0) \in E_R^\times \times \mathsf{G}_0(R)$, $(X, z) \in E_R^q \times E_R$, let us define $(q+2)\times(q+2)$-matrices

$$\mathsf{m}(t\,;g_0) := \begin{bmatrix} t & & \\ & g_0 & \\ & & (t^\iota)^{-1} \end{bmatrix}, \quad \mathsf{n}(X\,;z) := \begin{bmatrix} 1 & -X^* S_0 & z \\ & \mathtt{I}_q & X \\ & & 1 \end{bmatrix}.$$

Then there exists a unique closed $\mathbb{Q}$-subgroup M (resp. N) of G such that $\mathsf{M}(R)$ (resp. $\mathsf{N}(R)$) is the group formed by all the matrices $\mathsf{m}(t\,;g_0)$ with $(t, g_0) \in E_R^\times \times \mathsf{G}_0(R)$ (resp. $\mathsf{n}(X\,;z)$ with $(X, z) \in E_R^q \times E_R$ such that $z^\iota + z + S_0[X] = 0$). Then $\mathsf{P} := \mathsf{MN}$ is a parabolic $\mathbb{Q}$-subgroup of G with its unipotent radical N. The positive half-integer ρ_0 is defined by the relation $\det(\mathrm{Ad}(\mathsf{m}(t\,;\mathtt{I}_q))|\mathrm{Lie}(\mathsf{N})) = t^{2\rho_0}$ $(\forall t \in \mathbb{Q}^\times)$; explicitly $\rho_0 = 2^{-1}\delta q + \delta - 1$.

For a prime p, the stabilizer K_p in $\mathsf{G}(\mathbb{Q}_p)$ of the $\mathscr{O}_p$-lattice $\mathscr{L}_p := \mathscr{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ yields a maximal compact subgroup of $\mathsf{G}(\mathbb{Q}_p)$. Then the direct product $K_{\mathrm{f}} := \prod_p K_p$ is an open compact subgroup of $\mathsf{G}(\mathbb{A}_{\mathrm{f}})$.

The stabilizer K_∞ in $\mathsf{G}(\mathbb{R})$ of the line $\mathsf{v}_0^- E_{\mathbb{R}}$ with $\mathsf{v}_0^- := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0_q \\ -1 \end{bmatrix}$ yields a maximal compact subgroup of $\mathsf{G}(\mathbb{R})$. Note that $S[\mathsf{v}_0^-] = -1$. Put $\|\mathsf{v}\|_S := (S[\mathsf{v}] + 2\|S(\mathsf{v}, \mathsf{v}_0^-)\|^2)^{1/2}$, $(\mathsf{v} \in E_{\mathbb{R}}^{q+2})$ with $\|a\| := (\mathrm{N}(a))^{1/2}$ for $a \in E_{\mathbb{R}}$. Then $\|k\mathsf{v}\|_S = \|\mathsf{v}\|_S$ for $(\mathsf{v}, k) \in E_{\mathbb{R}}^{q+2} \times K_\infty$.

Corresponding to the cases (i), (ii) and (iii), $\mathsf{G}(\mathbb{R})/K_\infty$ is respectively isomorphic to (i) $\mathrm{O}(q+1, 1)/(\mathrm{O}(q+1) \times \mathrm{O}(1))$, (ii) $\mathrm{U}(q+1, 1)/(\mathrm{U}(q+1) \times \mathrm{U}(1))$, (iii) $\mathrm{Sp}(q+1, 1)/(\mathrm{Sp}(q+1)^c \times \mathrm{Sp}(1)^c)$ with $\mathrm{Sp}(l)^c$ the compact real form of the symplectic group of absolute rank l.

For $\mathsf{a} \in E^q$, put $\eta(\mathsf{a}) := \begin{bmatrix} 1 \\ \mathsf{a} \\ 1 \end{bmatrix} \in E^{q+2}$, and $\Delta(\mathsf{a}) := S[\eta(\mathsf{a})] = 2 + S_0[\mathsf{a}]$. Given a vector $\eta \in E^{q+2}$, its stabilizer in G is denoted by $\mathsf{G}^{[\eta]}$. If $\eta = \eta(\mathsf{a})$, then $\mathsf{G}^{[\eta]}(\mathbb{R})$ is a real-rank-one Lie group of the same type as $\mathsf{G}(\mathbb{R})$ but with one smaller matrix size than that of $\mathsf{G}(\mathbb{R})$. The intersection $\mathsf{G}^{[\eta]}(\mathbb{R}) \cap K_\infty$ is a maximal compact subgroup of $\mathsf{G}^{[\eta]}(\mathbb{R})$, since $S(\mathsf{v}_0^-, \eta(\mathsf{a})) = 0$.

3. Certain integral of Eisenstein series

In this section, after we fix Haar measures on various groups and recall the definition of the spherical Eisenstein series, we prove the convergence of the integral of the Eisenstein series restricted to $\mathsf{H}(\mathbb{Q})\backslash\mathsf{H}(\mathbb{A})$ with $\mathsf{H}=\mathsf{G}^{[\eta]}$.

3.1. Normalized Haar measures

For any closed $\mathbb{Q}$-subgroup N' of N, we normalize the Haar measure of $\mathsf{N}'(\mathbb{A})$ so that $\mathrm{mes}(\mathsf{N}'(\mathbb{Q})\backslash\mathsf{N}'(\mathbb{A}))=1$. Let us construct the Haar measure of $\mathsf{N}(\mathbb{A})$ explicitly. For each prime p, let $dx_p=d_{E_p}x_p$ be the Haar measures of E_p with $\int_{\mathcal{O}_p}dx_p=1$. Let $dx_\infty=d_{E_\mathbb{R}}x$ be $2^{\delta/2}D_{\mathcal{O}}^{-1/2}$-times the Euclidean measure of the space $E_\mathbb{R}$ with the norm $\|a\|$. Let $dx=d_{E_\mathbb{A}}x$ be the product measure of those dx_v's. Then $\mathrm{mes}(E\backslash E_\mathbb{A})=1$. Put the q-fold product measure $dX=dx^{\otimes q}$ and $dX_v=dx_v^{\otimes q}$ on $E_\mathbb{A}^q$ and E_v^q respectively. Since $E/E^{(0)}\cong\mathbb{Q}$, we can take the Haar measure $dy_v=d_{E_v^{(0)}}y_v$ of $E_v^{(0)}$ such that

$$(3.1)\qquad \int_{E_v} f(x)d_{E_v}x_v=|2|_v\int_{\mathbb{Q}_v}dt_v\int_{E_v^{(0)}}f(t_v+y_v)d_{E^{(0)}}y_v\,,\quad \forall f\in L^1(E_v)\,.$$

Let $dy=d_{E_\mathbb{A}^{(0)}}y$ be the product measure on $E_\mathbb{A}^{(0)}$ of dy_v's. Then if du_v denotes the Haar measure on $\mathsf{N}(\mathbb{Q}_v)$ such that

$$\int_{\mathsf{N}(\mathbb{Q}_v)}f(u_v)du_v=\int_{E_v^q}dX_v\int_{E_v^{(0)}}f(\mathsf{n}(X_v\,;\,y_v-2^{-1}S_0[X_v]))dy_v\,,\quad \forall f\in L^1(\mathsf{N}(\mathbb{Q}_v))\,,$$

then the du on $\mathsf{N}(\mathbb{A})$ is obtained as the product measure of du_v's.

LEMMA 1. *Let p be a prime and $\mathscr{A}$ a $\mathbb{Z}_p$-submodule of E_p.*

(1) *Let $t\in\mathbb{Q}_p$. Then the set $(t+\mathscr{A})\cap E^{(0)}\neq\varnothing$ if and only if $t\in 2^{-1}\tau(\mathscr{A})$ and in that case* $\mathrm{mes}((t+\mathscr{A})\cap E_p^{(0)})=\mathrm{mes}(\mathscr{A}\cap E_p^{(0)})$.

(2) *We have* $\mathrm{mes}(E_p^{(0)}\cap\mathscr{A})=\dfrac{\mathrm{mes}(\mathscr{A})}{\mathrm{mes}(\tau(\mathscr{A}))}$.

Let dk be the Haar measure of $K_\mathbb{A}:=K_\infty K_\mathrm{f}$ with $\mathrm{mes}(K_\mathbb{A})=1$. We have $\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})\cong\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1\times\mathbb{R}_+^\times$ with $\mathsf{M}(\mathbb{A})^1:=\{m\in\mathsf{M}(\mathbb{A})|\,|\det(\mathrm{Ad}(m)|\mathrm{Lie}(\mathsf{N})_\mathbb{A})|_\mathbb{A}=1\}$. Since $\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1$ is compact we have the Haar measure d^1m of $\mathsf{M}(\mathbb{A})^1$ so that $\mathrm{mes}(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1)=1$. Let $d_{\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})}\dot g$ be the $\mathsf{G}(\mathbb{A})$-invariant measure on $\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$ such that

$$\begin{aligned}&\int_{\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})}f(g)\,d_{\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})}\dot g\\ (3.2)\qquad &=\int_{\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1}d^1\dot m\int_0^\infty r^{-2\rho_0}\frac{dr}{r}\int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})}d\dot u\int_{K_\mathbb{A}}f(m\,u\,\mathsf{m}(r\,;\,\mathtt{I}_q)\,k)\,dk\,,\\ &\qquad\qquad\forall f\in L^1(\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))\,.\end{aligned}$$

We fix the Haar measure dg on $\mathsf{G}(\mathbb{A})$ such that $d_{\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})}\dot g$ is the associated quotient of dg. Then dg in turn yields an invaraint measure on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$, denoted by $d\dot g$.

For each $p \in I(\mathbb{Q})$, let $d\mu_p(x_p)$ be the $\mathsf{G}(\mathbb{Q}_p)$-invariant measure of $X_p := \mathsf{H}(\mathbb{Q}_p)\backslash \mathsf{G}(\mathbb{Q}_p)$ such that $\mathrm{mes}(\mathsf{H}(\mathbb{Q}_p)\backslash\mathsf{H}(\mathbb{Q}_p)K_p) = 1$. Let $d\mu_\infty(x_\infty)$ be the $\mathsf{G}(\mathbb{R})$-invariant measure on $X_\mathbb{R} := \mathsf{H}(\mathbb{R})\backslash\mathsf{G}(\mathbb{R})$ such that

$$\int_{X_\mathbb{R}} f(x_\infty)\, d\mu_\infty(x_\infty) = \int_{K_\infty} dk_\infty \int_0^\infty f(a_t\, k_\infty)(\sinh t)^{\delta-1}(\cosh t)^{2\rho_0-\delta+1}dt, \quad \forall f \in L^1(X_\mathbb{R})$$

([8, Theorem 2.5 (p. 110)]). Since $X_\mathbb{A}$ is a restricted direct product of the spaces X_v ($v \in I(\mathbb{Q}) \cup \{\mathbb{R}\}$), we have the product measure $d\mu_{X_\mathbb{A}}(x) := \bigotimes_v' d\mu_v$ on $X_\mathbb{A}$, which is $\mathsf{G}(\mathbb{A})$-invariant.

3.2. Eisenstein series and its integral

The double coset space $C_E := E^\times\backslash E_\mathbb{A}^\times/\mathbb{R}_+^\times U_\mathbb{A}$ with $U_\mathbb{A} := E_\mathbb{R}^{(1)} \times \prod_p \mathscr{O}_p^\times$ is a finite set. Since G_0 is $\mathbb{Q}$-anisotropic, the double coset space $C_{S_0} := \mathsf{G}_0(\mathbb{Q})\backslash\mathsf{G}_0(\mathbb{A})/K_{0,\mathbb{A}}$ with $K_{0,\mathbb{A}} := \{g_0 \in \mathsf{G}_0(\mathbb{A}) |\, g_0\mathscr{O}_p^q = \mathscr{O}_p^q\ (\forall p \in I(\mathbb{Q}))\}$ is also a finite set.

By the Iwasawa decomposition $\mathsf{G}(\mathbb{A}) = \mathsf{M}(\mathbb{A})\mathsf{N}(\mathbb{A})K_\mathbb{A}$, an element $g \in \mathsf{G}(\mathbb{A})$ can be written as $g = \mathsf{m}(t(g)\,;[g]_0)\,u\,\mathsf{k}(g)$ with $u \in \mathsf{N}(\mathbb{A})$, $[g]_0 \in \mathsf{G}_0(\mathbb{A})$ and $t(g) \in E_\mathbb{A}^\times$. It is easy to see that the cosets $t(g)U_\mathbb{A}$ and $[g]_0K_{0,\mathbb{A}}$ are determined uniquely by g.

To functions $\psi : C_E \to \mathbb{C}$ and $\xi : C_{S_0} \to \mathbb{C}$, one can attach the spherical Eisenstein series

$$E(\nu\,; \psi, \xi : g) := \sum_{\gamma\in\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})} \psi(t(\gamma g))\xi([\gamma g]_0)|\mathrm{N}(t(\gamma g))|_\mathbb{A}^{(\nu+\rho_0)/2}, \quad \nu \in \mathbb{C},\ g \in \mathsf{G}(\mathbb{A})\,,$$

which is absolutely convergent for ν with $\mathrm{Re}(\nu) > \rho_0$ and is defined by means of a meromorphic continuation for ν outside the domain of convergence ([15, IV 1. 9, Proposition (p. 140)]).

We recall the definition of the truncation operator ([2]) for later use. For $0 \leqslant T' \leqslant T \leqslant +\infty$, let $\chi_{[T',T]} : E_\mathbb{A}^\times \to \{0, 1\}$ be the characteristic function of $\{t \in E_\mathbb{A}^\times |\, |\mathrm{N}(t)|_\mathbb{A}^{1/2} \in [T', T]\}$. Then the truncation of a continuous function f on $\mathsf{G}(\mathbb{Q})\backslash \mathsf{G}(\mathbb{A})$ by $T > 0$ is defined by the formula

(3.3)
$$\bigwedge^T f(g) := f(g) - \sum_{\gamma\in\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})} \left(\int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})} f(u\gamma g)d\dot{u}\right)\chi_{[T,+\infty]}(t(\gamma g))\,,\ g \in \mathsf{G}(\mathbb{A})\,.$$

The action of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} := \mathrm{Lie}(\mathsf{G}(\mathbb{R}))_\mathbb{C}$ from the right on the smooth functions on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$ is denoted by R.

PROPOSITION 1. *Let $B \subset \{\nu \in \mathbb{C} |\, |\mathrm{Re}(\nu)| < \rho_0 - \delta\}$ be a compact set disjoint from the poles of $E(\nu\,; \psi, \xi)$ and C a compact subset of $\mathsf{G}(\mathbb{A})$. Let $D \in U(\mathfrak{g})$. Then the integral*

$$(3.4) \qquad \mathcal{P}(\nu\,; \psi, \xi : g : D) := \int_{\mathsf{H}(\mathbb{Q})\backslash\mathsf{H}(\mathbb{A})} R_D E(\nu\,; \psi, \xi : hg)d\dot{h}\,, \quad (\nu, g) \in B \times C$$

converges absolutely and uniformly on $B \times C$.

Proof. It is sufficient to prove the proposition under the assumption that the intersection $\mathsf{P}_\mathsf{H} := \mathsf{H} \cap \mathsf{P}$ is a parabolic $\mathbb{Q}$-subgroup of H with its unipotent radical $\mathsf{N}_\mathsf{H} :=$

$\mathsf{H} \cap \mathsf{N}$. We write $E(\nu)$ for $E(\nu\,;\psi,\xi)$ shortly. Let ω be the fundamental domain of $\mathsf{P}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})\mathsf{M}(\mathbb{A})^1$ and put $\omega_{\mathsf{H}} := \omega \cap \mathsf{H}(\mathbb{A})$. Then by reduction theory, there exists a positive number T_0 such that $\mathfrak{S}_{\mathsf{H}}^{T_0} := \omega_{\mathsf{H}} \cdot \{\mathsf{m}(t\,;\mathtt{1}_q) \,|\, t > T_0\} \cdot (K_{\mathbb{A}} \cap \mathsf{H}(\mathbb{A}))$ gives a fundamental domain of $\mathsf{H}(\mathbb{Q})\backslash\mathsf{H}(\mathbb{A})$ in a weak sense. In particular we have

$$\int_{\mathsf{H}(\mathbb{Q})\backslash\mathsf{H}(\mathbb{A})} |R_D E(\nu : hg)| \, d\dot{h} = \int_{\mathfrak{S}_{\mathsf{H}}^{T_0}} |R_D E(\nu : hg)| \, dh \,. \tag{3.5}$$

For $T > 0$, put $\mathfrak{S}^T := \omega \cdot \{\mathsf{m}(t\,;\mathtt{1}_q) \,|\, t > T\} \cdot K_{\mathbb{A}}$. Then we can find $T' > T_0$ sufficiently large such that $\gamma \in \mathsf{G}(\mathbb{Q})$ satisfies $\gamma\mathfrak{S}^{T'} \cap \mathsf{N}(\mathbb{A})\mathfrak{S}^{T'} \neq \varnothing$ if and only if $\gamma \in \mathsf{P}(\mathbb{Q})$ ([3, 17.9 Proposition (p. 120)]). Consequently we have

$$\bigwedge^{T} R_D E(\nu : g) = R_D E(\nu : g) - R_D E(\nu : g)_{(\mathsf{P})} \,, \quad g \in \mathsf{N}(\mathbb{A})\mathfrak{S}^T,\ T \geqslant T' \tag{3.6}$$

where the constant term $E(\nu : g)_{(\mathsf{P})}$ is given by

$$E(\nu : g)_{(\mathsf{P})} = f_{(\nu)}(g) + M(\nu) f_{(-\nu)}(g) \,, \quad g \in \mathsf{G}(\mathbb{A}) \,, \tag{3.7}$$

$$f_{(\nu)}(g) := \psi(t(g))\xi([g]_0)|\mathrm{N}(t(g))|_{\mathbb{A}}^{(\nu+\rho_0)/2} \tag{3.8}$$

with a meromorphic function $M(\nu)$, which as well as $R_D E(\nu)$ and $\bigwedge^T R_D E(\nu)$ are holomorphic on B. Since C is compact we can take $T > T'$ such that $\mathfrak{S}^T C \subset \mathsf{N}(\mathbb{A})\mathfrak{S}^{T'}$. The integral (3.5) is dominated by a sum of two integrals $I_1 := \int_{\mathfrak{S}_{\mathsf{H}}^{T_0}} |\bigwedge^T R_D E(\nu : hg)| \, dh$ and $I_2 := \int_{\mathfrak{S}_{\mathsf{H}}^{T_0}} |R_D E(\nu : hg) - \bigwedge^T R_D E(\nu : hg)| \, dh$. Since $\bigwedge^T R_D E(\nu)$ is rapidly decreasing on $\mathsf{G}(\mathbb{A})$, a fortiori bounded, the first integral I_1 is finite. We have $\mathfrak{S}_{\mathsf{H}}^T C \subset \mathfrak{S}^T C \subset \mathsf{N}(\mathbb{A})\mathfrak{S}^{T'}$. Hence by using (3.6) and (3.7), the second integral I_2 with $g \in C$ is dominated by

$$\begin{aligned}
&\int_{\mathfrak{S}_{\mathsf{H}}^{T_0} - \mathfrak{S}_{\mathsf{H}}^{T}} \left| R_D E(\nu : hg) - \bigwedge^{T} R_D E(\nu : hg) \right| dh + \int_{\mathfrak{S}_{\mathsf{H}}^{T}} \left| R_D E(\nu : hg) - \bigwedge^{T} R_D E(\nu : hg) \right| dh \\
&\qquad \leqslant \int_{\mathfrak{S}_{\mathsf{H}}^{T_0} - \mathfrak{S}_{\mathsf{H}}^{T}} \left| R_D E(\nu : hg) - \bigwedge^{T} R_D E(\nu : hg) \right| dh \\
&\qquad\quad + \int_{\mathfrak{S}_{\mathsf{H}}^{T}} \left| R_D f_{(\nu)}(hg) \right| dh + |M(\nu)| \int_{\mathfrak{S}_{\mathsf{H}}^{T}} \left| R_D f_{(-\nu)}(hg) \right| dh \,.
\end{aligned}$$

In the right-hand side, the first term is finite because $\mathfrak{S}_{\mathsf{H}}^{T_0} - \mathfrak{S}_{\mathsf{H}}^{T}$ is relatively compact. Let $k_\infty(g)$ be the Archimedean component of $\mathsf{k}(g)$. Then since $\mathrm{Ad}(K_\infty)$ preserves the natural degree-filtration of $U(\mathfrak{g})$, there exists a finite family $\{D_j\}_{j\in J}$ in $U(\mathfrak{g})$ and a family of smooth bounded functions $\{a_j\}_{j\in J}$ on $\mathsf{G}(\mathbb{A}) \times \mathsf{G}(\mathbb{A})$ such that $\mathrm{Ad}(k_\infty(k_\infty(h)g))D = \sum_{j\in J} a_j(h\,;g)D_j$ for $(h, g) \in \mathsf{G}(\mathbb{A}) \times \mathsf{G}(\mathbb{A})$. By the Iwasawa decomposition $\mathsf{G}(\mathbb{R}) = \mathsf{P}(\mathbb{R})K_\infty$, we may assume that D_j is of the form $D'_j Y^{m_j} D''_j$ with $D'_j \in U(\mathrm{Lie}(\mathsf{N}(\mathbb{R})),\ D''_j \in U(\mathrm{Lie}(K_\infty)),\ m_j \in \mathbb{N}$ and with Y the differential operator $t\frac{d}{dt}$ on the torus $\{\mathsf{m}(t\,;\mathtt{1}_q) \,|\, t > 0\}$. Noticing that $R_D f_{(\nu)}$ is left $\mathsf{N}(\mathbb{A})\mathsf{G}_0(\mathbb{A})$-invariant and right K_{f}-invariant, we have

$$\begin{aligned}|(R_D f_{(\nu)})(hg)| &= |(R_D f_{(\nu)})\big(\mathsf{m}(t(h)\,t(\mathsf{k}(h)g)\,;\,\mathtt{I}_q)\,\mathsf{k}(\mathsf{k}(h)g)\big)| \\ &= \Big|\sum_{j\in J} a_j(h\,;\,g)\,R_{D_j} f_{(\nu)} l(\mathsf{m}(t(h)\,t(\mathsf{k}(h)g)\,;\,\mathtt{I}_q))\Big| \\ &\leqslant L_0 \Big|\sum_{j\in J'} a_j(h\,;\,g)\,(\nu+\rho_0)^{m_j}\,|\mathrm{N}(t(h)\,t(\mathsf{k}(h)g))|_{\mathbb{A}}^{(\nu+\rho_0)/2}\Big|\,.\end{aligned}$$

Here $L_0 = \sup\{|\xi(g_0)|\,|\, g_0 \in \mathsf{G}_0(\mathbb{A})\}$ and J' is the set of indices $j \in J$ with $D'_j = D''_j = 1$. Since $K_{\mathbb{A}}C$ is compact, this gives an estimation of the form

$$|R_D f_{(\nu)}(hg)| \leqslant L_{B,C}\,|\mathrm{N}(t(h))|_{\mathbb{A}}^{(\mathrm{Re}(\nu)+\rho_0)/2}, \quad \forall(\nu,h,g) \in B\times\mathfrak{S}_{\mathsf{H}}^T\times C\,.$$

with some constant $L_{B,C} > 0$. Using this estimate and the integration formula associated with the Iwasawa coordinate on $\mathfrak{S}_{\mathsf{H}}^T$, we can estimate $\int_{\mathfrak{S}_{\mathsf{H}}^T} |R_D f_{(\pm\nu)}(hg)|\,dh \leqslant L_{B,C}\mathrm{mes}(\omega_{\mathsf{H}}) \int_T^\infty t^{\pm\mathrm{Re}(\nu)-\rho_0+\delta}\frac{dt}{t}$. By the assumption $|\mathrm{Re}(\nu)| < \rho_0 - \delta$, this integral is finite. □

3.3. Unramified Grössencharacters for $E^\times$

The product $\mathscr{O}_{\mathrm{f}}^\times = \prod_p \mathscr{O}_p^\times$ is a maximal compact subgroup of $E_{\mathrm{f}}^\times := \prod'_p E_p^\times$. For each $p \in I(\mathbb{Q})$, let $d^\times t_p$ be the Haar measure of $E_p^\times$ such that $\mathrm{mes}(\mathscr{O}_p^\times) = 1$. Then we have the product measure $d^\times t_{\mathrm{f}} = \bigotimes'_p d^\times t_p$ on $E_{\mathrm{f}}^\times$. The Hecke algebra $\mathscr{H}_{\mathrm{f}} := \mathscr{H}(E_{\mathrm{f}}^\times, \mathscr{O}_{\mathrm{f}}^\times, d^\times t_{\mathrm{f}})$, which is a restricted tensor product of local Hecke algebras $\mathscr{H}_p := \mathscr{H}(E_p^\times, \mathscr{O}_p^\times, d^\times t_p)$, is commutative and acts fully-reducibly from right on the finite dimensional vector space of $\mathbb{C}$-valued functions on C_E by convolution on $E_{\mathrm{f}}^\times$. Let $\hat{C}_E$ be the set of functions ψ on C_E with $\psi(1) = 1$ and with the property that there exists a $\mathbb{C}$-algebra homomorphism $\boldsymbol{\lambda}_\psi : \mathscr{H}_{\mathrm{f}} \to \mathbb{C}$ such that

$$\int_{E_{\mathrm{f}}^\times} \psi(xy_{\mathrm{f}})\phi(y_{\mathrm{f}})\,d^\times y_{\mathrm{f}} = \boldsymbol{\lambda}_\psi(\phi)\,\psi(x)\,, \quad \forall x \in E_{\mathbb{A}}^\times,\ \forall \phi \in \mathscr{H}_{\mathrm{f}}\,. \tag{3.9}$$

We note that $\boldsymbol{\lambda} : \mathscr{H}_{\mathrm{f}} \to \mathbb{C}$ uniquely determine a family of algebra homomorphisms $\lambda^{(p)} : \mathscr{H}_p \to \mathbb{C}$ such that $\boldsymbol{\lambda}(\phi) = \prod_p \lambda^{(p)}(\phi^{(p)})$ for any decomposable function $\phi = \prod_p \phi^{(p)} \in \mathscr{H}_{\mathrm{f}}$. (We call $\lambda^{(p)}$ the p-component of $\boldsymbol{\lambda}$.)

LEMMA 2. *For $\psi \in \hat{C}_E$, put $\tilde{\psi}(x) = \int_{\mathscr{O}_{\mathrm{f}}^\times} \psi(u_{\mathrm{f}}x)\,d^\times u_{\mathrm{f}}$, $x \in E_{\mathbb{A}}^\times$. Then*

$$\tilde{\psi}(x) = \frac{\boldsymbol{\lambda}_\psi(\mathrm{Char}_{\mathscr{O}_{\mathrm{f}}^\times x_{\mathrm{f}}\mathscr{O}_{\mathrm{f}}^\times})}{\mathrm{mes}(\mathscr{O}_{\mathrm{f}}^\times x_{\mathrm{f}}\mathscr{O}_{\mathrm{f}}^\times)} = \prod_p \frac{\lambda_\psi^{(p)}(\mathrm{Char}_{\mathscr{O}_p^\times x_p\mathscr{O}_p^\times})}{\mathrm{mes}(\mathscr{O}_p^\times x_p\mathscr{O}_p^\times)}\,, \quad \forall x = (x_v) \in E_{\mathbb{A}}^\times\,. \tag{3.10}$$

Here x_{f} denotes the finite component of $x \in E_{\mathbb{A}}^\times$.

Proof. From (3.9), we have $\int_{E_{\mathrm{f}}^\times} \tilde{\psi}(y_{\mathrm{f}})\phi(y_{\mathrm{f}})\,d^\times y_{\mathrm{f}} = \boldsymbol{\lambda}_\psi(\phi)\,\tilde{\psi}(1)$. To obtain the formula (3.10), just put $\phi = \mathrm{Char}_{\mathscr{O}_{\mathrm{f}}^\times x_{\mathrm{f}}\mathscr{O}_{\mathrm{f}}^\times}$ in this equation and note $\tilde{\psi}$ is bi-$\mathscr{O}_{\mathrm{f}}^\times$-invariant. □

Recall that the L-series for $\boldsymbol{\lambda}$ is defined to be the Euler product of local factors $L^{(p)}(\nu\,;\lambda^{(p)})$ given as follows.

- If E_p is division, then $\mathscr{H}_p$ is a polynomial algebra of one variables $\mathscr{H}_p = \mathbb{C}[T]$ with $T = \mathrm{Char}_{\mathscr{O}_p^\times \varpi \mathscr{O}_p^\times}$. Put $L^{(p)}(\nu\,;\lambda^{(p)}) := (1-\lambda^{(p)}(T)|\mathrm{N}(\varpi)|_p^{\nu})^{-1}$.
- If $E_p = \mathbb{Q}_p \oplus \mathbb{Q}_p$, then $\mathscr{H}_p$ is a polynomial algebra of two variable $\mathbb{C}[T', T'']$ with $T' = \mathrm{Char}_{\mathscr{O}_p^\times \varpi_1 \mathscr{O}_p^\times}$, $T'' = \mathrm{Char}_{\mathscr{O}_p^\times \varpi_2 \mathscr{O}_p^\times}$ ($\varpi_1 = (p, 1)$, $\varpi_2 = (1, p)$). Put $L^{(p)}(\nu\,;\lambda^{(p)}) := (1-\lambda^{(p)}(T')p^{-\nu})^{-1}(1-\lambda^{(p)}(T'')p^{-\nu})^{-1}$.
- If $E_p = \mathrm{M}_2(\mathbb{Q}_p)$ and $\mathscr{O}_p = \mathrm{M}_2(\mathbb{Z}_p)$, then $\mathscr{H}_p$ is a polynomial algebra of two variables $\mathbb{C}[T_1, T_0]$ with $T_1 = \mathrm{Char}_{\mathscr{O}_p^\times \left[\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right] \mathscr{O}_p^\times}$, $T_0 = \mathrm{Char}_{\mathscr{O}_p^\times \left[\begin{smallmatrix} p & 0 \\ 0 & p\end{smallmatrix}\right] \mathscr{O}_p^\times}$. Put
$$L^{(p)}(\nu\,;\lambda^{(p)}) := (1-\lambda^{(p)}(T_1)p^{-\nu} + \lambda^{(p)}(T_0)p^{1-2\nu})^{-1}\,.$$

Let $\psi \in \hat{C}_E$. Then it is known that the Euler product $L(\nu\,;\boldsymbol{\lambda}_\psi)$ is absolutely convergent on $\mathrm{Re}(\nu) > \delta$ and has an meromorphic continuation to the whole $\mathbb{C}$.

LEMMA 3. *Let* $\psi \in \hat{C}_E$. *The each* p-*component* $\lambda_\psi^{(p)}$ *of* $\boldsymbol{\lambda}_\psi$ *satisfies* $\lambda_\psi^{(p)}(\mathrm{Char}_{\mathscr{O}_p^\times p \mathscr{O}_p^\times}) = 1$.

Proof. By the decomposition $\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_+^\times \prod_p \mathbb{Z}_p^\times$, we have $\psi(xa) = \psi(x)$ ($\forall x \in E_\mathbb{A}^\times$, $a \in \mathbb{A}^\times$) on one hand. On the other hand, the equation (3.9) gives $\psi(xa_\mathrm{f}) = \boldsymbol{\lambda}_\psi(\mathrm{Char}_{\mathscr{O}_\mathrm{f}^\times a_\mathrm{f} \mathscr{O}_\mathrm{f}^\times})\psi(x)$. □

REMARK.
1. When E is abelian, the set C_E is a finite abelian group and $\hat{C}_E$ is just its Pontrjagin dual. We also note $\hat{C}_\mathbb{Q} = \{1\}$.
2. $\zeta_\mathscr{O}(\nu) := L(\nu\,;1)$ is the zeta-function of the maximal order $\mathscr{O}$.

LEMMA 4. *Assume* $E_p = \mathrm{M}_2(\mathbb{Q}_p)$ *and* $\mathscr{O}_p = \mathrm{M}_2(\mathbb{Z}_p)$. *For a* $\mathbb{C}$-*algebra homomorphism* $\lambda^{(p)} : \mathscr{H}_p \to \mathbb{C}$ *and* $\nu \in \mathbb{C}$,
$$\begin{aligned} h(\nu\,;\lambda^{(p)}) &:= \sum_{k=0}^{\infty} \lambda^{(p)}\left(\mathrm{Char}_{\mathscr{O}_p^\times \left[\begin{smallmatrix} p^k & 0 \\ 0 & 1\end{smallmatrix}\right] \mathscr{O}_p^\times}\right) p^{-k\nu} \\ &= (1-\lambda^{(p)}(\mathrm{Char}_{\mathscr{O}_p^\times p \mathscr{O}_p^\times})p^{-2\nu})\, L^{(p)}(\nu\,;\lambda^{(p)})\,. \end{aligned}$$

Proof. This follows from [4, Proposition 4.6.4 (p. 494)] after some elementary computation. □

4. Certain Poincaré series and its properties

From now on we fix $\mathsf{a} \in S_0^{-1}\mathscr{O}^q$. Put $\eta = \eta(\mathsf{a})$, $\Delta = \Delta(\mathsf{a})$ and $\mathsf{H} = \mathsf{G}^{[\eta]}$.

4.1. Definition of Poincare series

Put $\mathsf{v}_\eta^+ := \Delta^{-1/2}\eta \in E_\mathbb{R}^{q+2}$; then $S[\mathsf{v}_\eta^+] = +1$, $S_0(\mathsf{v}_0^-, \mathsf{v}_\eta^+) = 0$. For $t \in \mathbb{R}$, there exists a unique element $a_t \in \mathsf{G}(\mathbb{R})$ acting trivially on the orthogonal complement of the plane $\mathsf{v}_\eta^+ E_\mathbb{R} + \mathsf{v}_0^- E_\mathbb{R}$ and satisfying the formula

$$a_t(\mathsf{v}_\eta^+) = \cosh t \cdot \mathsf{v}_\eta^+ + \sinh t \cdot \mathsf{v}_0^-, \quad a_t(\mathsf{v}_0^-) = \sinh t \cdot \mathsf{v}_\eta^+ + \cosh t \cdot \mathsf{v}_0^- .$$

$A_\infty := \{a_t | \, t \in \mathbb{R}\}$ is an $\mathbb{R}$-split torus in $\mathsf{G}(\mathbb{R})$. Let $\tilde{H}_\infty$ be the stabilizer in $\mathsf{G}(\mathbb{R})$ of the line $\mathsf{v}_\eta^+ E_\mathbb{R}$; it is isomorphic to the direct product $\mathsf{H}(\mathbb{R}) \times T$ with $T = \{g \in \mathsf{G}(\mathbb{R}) | \, g|(\mathsf{v}_\eta^+ E_\mathbb{R})^\perp = \mathrm{id}\} \cong E_\mathbb{R}^{(1)}$. Since T is contained in the centralizer of A_∞, the decomposition $\mathsf{G}(\mathbb{R}) = \tilde{H}_\infty A_\infty^+ K_\infty$ ([8, Theorem 2.4 (p. 108)]) immediately gives a similar one $\mathsf{G}(\mathbb{R}) = \mathsf{H}(\mathbb{R}) A_\infty^+ K_\infty$. Here $A_\infty^+ := \{a_t | \, t \geqslant 0\}$.

LEMMA 5. *Let $s \in \mathbb{C}$.*

(1) *There exists a unique C^∞-function $\phi_s^{(\infty)} : \mathsf{G}(\mathbb{R}) \to \mathbb{C}$ such that*

$$\phi_s^{(\infty)}(h_\infty a_t k_\infty) = \left(\frac{1}{\cosh t}\right)^{s+\rho_0}, \quad \forall (h_\infty, a_t, k_\infty) \in \mathsf{H}(\mathbb{R}) \times A_\infty \times K_\infty .$$

(2) *For a compact subset B of $\mathsf{G}(\mathbb{R}) \times \{s \in \mathbb{C} | \operatorname{Re}(s) > \rho_0\}$, there exist $C_B > 0$ and $\sigma > \rho_0$ such that*

$$|\phi_s^{(\infty)}(xg)| \leqslant C_B \cdot \phi_\sigma^{(\infty)}(x), \quad \forall (g, s) \in B, \ \forall x \in \mathsf{G}(\mathbb{R}) .$$

Proof. (1) This follows from $\mathsf{G}(\mathbb{R}) = \tilde{H}_\infty A_\infty K_\infty$ and [6, Theorem 4.1 (ii)].

(2) First note $\sigma \mapsto \phi_\sigma^{(\infty)}(g)$ is increasing in $\sigma \geqslant 0$ for a fixed g. We have only to show

$$\sup_{(x,g)\in \mathsf{G}(\mathbb{R})\times B_2} \frac{\phi_\sigma^{(\infty)}(xg)}{\phi_\sigma^{(\infty)}(x)} < \infty \tag{4.1}$$

for any compact set $B_2 \subset \mathsf{G}(\mathbb{R})$. The two vectors $\mathsf{f} := 2^{-1/2}(\mathsf{v}_\eta^+ + \mathsf{v}_0^-)$ and $\mathsf{f}' := 2^{-1/2}(\mathsf{v}_\eta^+ - \mathsf{v}_0^-)$ span a hyperbolic plane in $E_\mathbb{R}^{q+2}$. Correspondingly we have a parabolic subgroup Q of the Lie group $\mathsf{G}(\mathbb{R})$ stabilizing $\mathsf{f}E_\mathbb{R}$. Let U be the unipotent radical of Q. Let $t \geqslant 0, u \in U$. Then $u\mathsf{f} = \mathsf{f}$ and $u\mathsf{f}' = \mathsf{f}' - X + \mathsf{f}\,(y - 2^{-1}S[X])$ $(\exists y \in E_\mathbb{R}^{(1)}, \ \exists X \in (\mathsf{v}_0^- E_\mathbb{R} + \mathsf{v}_\eta^+ E_\mathbb{R})^\perp)$. We can decompose $a_t u$ as $a_t u = h a_r k$ with $h \in \mathsf{H}(\mathbb{R})$, $k \in K_\infty$ and $r \in \mathbb{R}$. Hence $\|(a_t u)^{-1}\eta\|_S^2 = 2\cosh^2 r - 1$ on one hand. On the other hand, since $S((a_t u)^{-1}\mathsf{v}_\eta^+, \mathsf{v}_0^-) = \sinh t - e^t(y - 2^{-1}S[X])$, we have $\|(a_t u)^{-1}\mathsf{v}_\eta^+\|_S^2 = 1 + 2\|\sinh t + e^t(2^{-1}S[X] - y)\|^2$. Therefore

$$\begin{aligned}(\cosh r)^2 &= 1 + \|\sinh t + e^t(2^{-1}S[X] - y)\|^2 \\ &= \cosh^2 t + e^t \sinh t \, S[X] + e^{2t}\|y\|^2 + 4^{-1}e^{2t}(S[X])^2 .\end{aligned}$$

From this we obtain $\cosh r \geqslant \cosh t$, noting $S[X] \geqslant 0$ and $\sinh t \geqslant 0$. Since $\sigma + \rho_0 > 0$, this in turn gives the inequality

$$\phi_\sigma^{(\infty)}(a_t u) = \phi_\sigma^{(\infty)}(a_r) \leqslant \phi_\sigma^{(\infty)}(a_t), \quad (\forall t \geqslant 0, \ \forall u \in U) . \tag{4.2}$$

Now let $g \in K_\infty B_2$ and write $g = a_{t_g} u_g k_g$ along the Iwasawa decomposition $\mathsf{G}(\mathbb{R}) = A_\infty U K_\infty$. Since $\{t_g | \, g \in K_\infty B_2\}$ is compact, there is $t_0 > 0$ such that $t + t_g \geqslant 0$ $(\forall t > t_0, \ \forall g \in K_\infty B_2)$. We have

$$\phi_\sigma^{(\infty)}(a_t g) = \phi_\sigma^{(\infty)}(a_{t+t_g} u_g) \leqslant \phi_\sigma^{(\infty)}(a_{t+t_g}) \leqslant C'' \phi_\sigma^{(\infty)}(a_t), \ \forall t > t_0, \ \forall g \in K_\infty B_2$$

with some constant C''. From this, (4.1) follows. □

LEMMA 6. *For* $D \in U(\mathfrak{g})$ *and a compact subset* B_1 *of* $\mathrm{Re}(s) > \rho_0$, *there exist* $C_{D,B_1} > 0$ *and* $\sigma > \rho_0$ *such that*

$$|R_D\phi_s^{(\infty)}(g)| \leqslant C_{D,B_1}|\phi_\sigma^{(\infty)}(g)|\,, \quad \forall g \in \mathsf{G}(\mathbb{R}),\ \forall s \in B_1\,.$$

Proof. Put $\mathfrak{a} = \mathrm{Lie}(A_\infty)_\mathbb{C}$, $\mathfrak{h} = \mathrm{Lie}(\mathsf{H}(\mathbb{R}))_\mathbb{C}$ and $\mathfrak{k} = \mathrm{Lie}(K_\infty)_\mathbb{C}$. Let $\mathfrak{n}^-$ be the $\mathbb{C}$-span of negative root vectors with respect to $\mathfrak{a}$. Then the decomposition $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{a} + \mathfrak{k}$ yields a direct sum decomposition $U(\mathfrak{g}) = (U(\mathfrak{a})U(\mathfrak{k}))\oplus\mathfrak{n}^-\, U(\mathfrak{g})$. Let D_0 be the projection of D to the first component $U(\mathfrak{a})U(\mathfrak{k})$. By a similar argument as that in the proof of [12, Lemma 8.27 (p. 229)], we can show that there exist $Z_j \in U(\mathfrak{k})$, $X_j \in U(\mathfrak{h})$, functions $f_j(t)$ on $t > 0$ and $m_j \in \mathbb{N}$ such that each $f_j(t)$ is a polynomial of $e^{-t}(\cosh t)^{-1}$, $(\cosh t)^{-1}$, $e^{-2t}(\sinh(2t))^{-1}$ and $(\sinh(2t))^{-1}$ and such that

$$D = D_0 + \sum_{j=1}^{p} f_j(t)(\mathrm{Ad}(a_t)^{-1}X_j)Y^{m_j}Z_j\,, \quad \forall t > 0\,. \tag{4.3}$$

Here $Y \in \mathfrak{a}$ is such that $a_t = \exp(tY)$. In particular $f_j(t)$ is bounded as $t \to \infty$. Since $\phi_s^{(\infty)}$ is left $\mathsf{H}(\mathbb{R})$-invariant and right K_∞-invariant, we have $R_{(\mathrm{Ad}(a_t)^{-1}X_j)Y^{m_j}Z_j}(a_t) = 0$ identically in $t > 0$ if $\deg(X_j) + \deg(Z_j) > 0$. Since $D_0 \in U(\mathfrak{a})\,U(\mathfrak{k})$, there exists a differential operator ∂_t with constant coefficients such that $R_{D_0}\phi_s^{(\infty)}(a_t) = \partial_t\phi_s^{(\infty)}(a_t)$. Noting this remark and using the expression (4.3), we can show the estimation

$$|R_D\phi_s^{(\infty)}(a_t)| \leqslant C_{D,B_1}\phi_\sigma^{(\infty)}(a_t)\,, \quad \forall t > 0,\ \forall s \in B_1 \tag{4.4}$$

with a constant C_{D,B_1} and some $\sigma = \sigma_D > \rho_0$. Since $U^j(\mathfrak{g})$ is $\mathrm{Ad}(K_\infty)$-invariant we can find finite number of elements D_j in $U(\mathfrak{g})$ and smooth functions $c_j(k)$ on K_∞ such that $\mathrm{Ad}(k)D = \sum_j c_j(k)\,D_j$ ($\forall k \in K_\infty$). Combining this remark with the relation $R_D\phi_s^{(\infty)}(ha_tk) = R_{\mathrm{Ad}(k)D}\phi_s^{(\infty)}(a_t)$ for $(h, a_t, k) \in \mathsf{H}(\mathbb{R}) \times A_\infty \times K_\infty$, we obtain the conclusion from the estimations (4.4) for D_j's. □

For a prime number p, let $\phi^{(p)} : \mathsf{G}(\mathbb{Q}_p) \to \mathbb{C}$ be the characteristic function of the open subset $\mathsf{H}(\mathbb{Q}_p)K_p$ of $\mathsf{G}(\mathbb{Q}_p)$. Then the formula

$$\phi_s(g) = \phi_s^{(\infty)}(g_\infty)\prod_p \phi^{(p)}(g_p)\,, \quad g = (g_v) \in \mathsf{G}(\mathbb{A})$$

defines a smooth function $\phi_s : \mathsf{G}(\mathbb{A}) \to \mathbb{C}$ on $\mathsf{G}(\mathbb{A})$, that is left $\mathsf{H}(\mathbb{A})$-invariant and right $K_\mathbb{A}$-invariant. Now let us introduce the Poincaré series

$$F_s(g) := \sum_{\gamma \in H(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})} \phi_s(\gamma g)\,, \quad g \in \mathsf{G}(\mathbb{A})\,.$$

PROPOSITION 2. *The series* $F_s(g)$ *converges absolutely and locally uniformly with respect to* $(g, s) \in \mathsf{G}(\mathbb{A}) \times \{s \in \mathbb{C}|\,\mathrm{Re}(s) > \rho_0\}$. *For* $\mathrm{Re}(s) > \rho_0$, *the function* $F_s : \mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}) \to \mathbb{C}$ *is smooth, uniformly of moderate growth and square-integrable.*

REMARK.

Let $\mathrm{Re}(s) > \rho_0$. Let Ω be the Casimir element of $\mathfrak{g}$ (see 7.1). Then the function F_s satisfies the differential equation (c.f., [13, Theorem 4.3 (ii) (p. 224)])

$$(R_\Omega + \rho_0^2 - s^2)F_s(g) = (\rho_0 + s)(\rho_0 - \delta - s)F_{s+2}(g)\,.$$

4.2. Proof of Proposition 2

We skech a proof assuming H is $\mathbb{Q}$-isotropic. For details see [18], [16]. Retain the notations and assumptions in the proof of Proposition 1. Fix $\sigma > \rho_0$. Noting that $\|a_t^{-1}\mathsf{v}_\eta^+\|_S \leqslant 2^{1/2}\cosh t$, we have the estimate

$$\phi_\sigma(g) \leqslant 2^{(\sigma+\rho_0)/2}\|g_\infty^{-1}\mathsf{v}_\eta^+\|_S^{-(\sigma+\rho_0)}\phi^{(\mathrm{f})}(g_{\mathrm{f}})\,, \quad g \in \mathsf{G}(\mathbb{A})$$

with $\phi^{(\mathrm{f})}$ the product of $\phi^{(p)}$'s. Fix a Siegel domain $\mathfrak{S}^T$ for $\mathsf{G}(\mathbb{A})$. Since the function $\|g_\infty^{-1}\mathsf{v}_\eta^+\|_S$ on $\mathsf{G}(\mathbb{R})$ is the 'gauge function' in the sense of [16, Definition 4.2.1], by the same way as [16, Lemma 4.3.3], we obtain the estimate:

(4.5)

$$\sum_{\gamma\in\mathsf{H}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})}\phi_\sigma(\gamma x)$$
$$\prec |\mathrm{N}(t(x))|_{\mathbb{A}}^{\rho_0}\int_{\mathsf{H}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})x\mathcal{N}_\mathbb{R}K_{\mathrm{f}}}\|g_\infty^{-1}\mathsf{v}_\eta^+\|_S^{-(\sigma+\rho_0)}\phi^{(\mathrm{f})}(g_{\mathrm{f}})\,dg\,, \forall x \in \mathfrak{S}^T\,.$$

Here $\mathcal{N}_\mathbb{R}$ is a neighborhood of identity in $\mathsf{G}(\mathbb{R})$. In order to bound the integral from above, we use the following lemma, which is a sort of adaptation of [18].

LEMMA 7. *Let $\mathfrak{S}^T$ and $\mathfrak{S}_{\mathsf{H}}^{T_0}$ be Siegel domains of G and H (with respect to P and P_{H}) respectively. Let $\mathfrak{B}$ be the set of those $(\gamma, x, k\,; h, a_t, k') \in \mathsf{G}(\mathbb{Q})\times\mathfrak{S}^T\times K_\mathbb{A}\times\mathfrak{S}_{\mathsf{H}}^{T_0}\times A_\infty\times K_\mathbb{A}$ such that $\gamma xk = ha_tk'$ in $\mathsf{G}(\mathbb{A})$. Then for some constant C we have*

$$|\mathrm{N}(t(x))|_{\mathbb{A}}^{-1/2} \geqslant C\cdot e^{-t}|\mathrm{N}(t(h))|_{\mathbb{A}}^{-1/2}\,, \quad \forall(\gamma, x, k\,; h, a_t, k') \in \mathfrak{B}\,.$$

Utilizing this lemma, as well as the estimate $\mathrm{mes}(\mathfrak{S}_{\mathsf{H}}^{T_0}) \prec T_0^{-2\rho_0+\delta}$, in the same way as [16, Proposition 4.3.1], we have

$$\int_{\mathsf{H}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})x\mathcal{N}_\mathbb{R}K_{\mathrm{f}}}\|g_\infty^{-1}\mathsf{v}_\eta^+\|_S^{-(\sigma+\rho_0)}\phi^{(\mathrm{f})}(g_{\mathrm{f}})\,dg \prec |\mathrm{N}(t(x))|_{\mathbb{A}}^{-\rho_0+\delta/2}\,, \quad x \in \mathfrak{S}^T\,. \tag{4.6}$$

Combining (4.5) and (4.6) together, we finally have

$$\sum_{\gamma\in\mathsf{H}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})}\phi_\sigma(\gamma x) \prec |\mathrm{N}(t(x))|_{\mathbb{A}}^{\delta/2}\,, \quad x \in \mathfrak{S}^T\,. \tag{4.7}$$

We easily obtain all assertions in Proposition 2 from (4.7), Lemma 5 and Lemma 6.

5. Constant term of F_s

In this section, s denotes a fixed complex number with $\mathrm{Re}(s) > \rho_0$.

5.1. Constant term and double coset spaces

The constant term of F_s along P is defined by the integral

$$F_{s,(\mathsf{P})}(g) := \int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})} F_s(ug)\,d\dot{u}\,. \tag{5.1}$$

LEMMA 8. *The constant term $F_{s,(\mathsf{P})}(g)$ equals the absolutely convergent sum of those integrals*

$$I_s(\gamma\,;\,g) := \int_{\mathsf{N}_\gamma(\mathbb{A})\backslash\mathsf{N}(\mathbb{A})} \phi_s(\gamma u g)d_\gamma\dot{u}$$

over all $\gamma \in \mathsf{H}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})/\mathsf{N}(\mathbb{Q})$. Here $\mathsf{N}_\gamma := \gamma^{-1}\mathsf{H}\gamma \cap \mathsf{N}$ and $d_\gamma\dot{u}$ is the quotient measure on $\mathsf{N}_\gamma(\mathbb{A})\backslash\mathsf{N}(\mathbb{A})$.

Proof. This can be proved by a standard argument. We note mes $(\mathsf{N}_\gamma(\mathbb{Q})\backslash\mathsf{N}_\gamma(\mathbb{A})) = 1$. □

By [14, Lemma 2.3], there exists a vector $x_0 \in E^q$ such that $S_0[x_0] = \Delta$. We fix such x_0 and put

$$\Upsilon_0 := \mathsf{w}_0\mathsf{n}(x_0 - \mathsf{a}\,;\,S_0(\mathsf{a}, x_0 - \mathsf{a}) - 1)\,, \quad \mathsf{w}_0 := \begin{bmatrix} & & 1 \\ & \mathtt{1}_q & \\ 1 & & \end{bmatrix}.$$

Then by [14, Proposition 2.4], $\mathsf{G}(\mathbb{Q}) = \mathsf{H}(\mathbb{Q})\mathsf{P}(\mathbb{Q}) \sqcup \mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{P}(\mathbb{Q})$.

LEMMA 9. *Let G_{0,x_0} be the stabilizer in G_0 of x_0. We have*

$$\mathsf{M}(\mathbb{Q}) \cap (\Upsilon_0\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{N}(\mathbb{Q})) = \{\mathsf{m}(t\,;\,g_0) \mid t \in E^\times,\ g_0 \in \mathsf{G}_{0,x_0}(\mathbb{Q})\}\,,$$
$$\Upsilon_0\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1} \cap \mathsf{N}(\mathbb{Q}) = \{\mathsf{n}(X\,;\,z) \in \mathsf{N}(\mathbb{Q}) \mid S_0(x_0, X) = 0\}\,.$$

Proof. A direct computation shows that the relation $\mathsf{m}(t;\,g_0)\mathsf{n}(X;\,z) \in \Upsilon_0\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}$ is equivalent to

$$g_0x_0 = x_0\,, \quad 1 - S_0(x_0, X) = t^{-1}\,. \tag{5.2}$$

This in particular shows that $\mathsf{m}(t\,;\,g_0) \in \Upsilon_0\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{N}(\mathbb{Q})$ implies $g_0 \in \mathsf{G}_{0,x_0}(\mathbb{Q})$, and also shows the second claim in our lemma. Conversely, for any element $\mathsf{m}(t\,;\,g_0) \in \mathsf{M}(\mathbb{Q})$ such that $g_0x_0 = x_0$, (5.2) is true with $X := \Delta^{-1}x_0(1 - (t^\iota)^{-1})$ and $z := -2^{-1}S_0[X]$. Hence we have $\mathsf{m}(t\,;\,g_0)\mathsf{n}(X\,;\,z) \in \Upsilon_0\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}$, especially $\mathsf{m}(t\,;\,g_0) \in \Upsilon_0\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{N}(\mathbb{Q})$. □

PROPOSITION 3. (1) *We have the disjoint union decomposition*

$$\mathsf{H}(\mathbb{Q})\mathsf{P}(\mathbb{Q}) = \bigsqcup_{t\in E^\times} \mathsf{H}(\mathbb{Q})\mathsf{m}(t\,;\,\mathtt{1}_q)\mathsf{N}(\mathbb{Q})\,.$$

(2) *We fix a complete set of representatives $\mathfrak{X}_\mathbb{Q}$ for the coset space $\mathsf{G}_{0,x_0}(\mathbb{Q})\backslash\mathsf{G}_0(\mathbb{Q})$. Then we have the disjoint union decomposition*

$$\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{P}(\mathbb{Q}) = \bigsqcup_{g_0\in\mathfrak{X}_\mathbb{Q}} \mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{m}(1\,;\,g_0)\mathsf{N}(\mathbb{Q})\,.$$

Proof. (1) It is obvious that $\mathsf{H}(\mathbb{Q})\mathsf{P}(\mathbb{Q})$ is a union of cosets $\mathsf{H}(\mathbb{Q})\mathsf{m}(t\,;g_0)\mathsf{N}(\mathbb{Q})$. For $(t, g_0) \in E^\times \times \mathsf{G}_0(\mathbb{Q})$, [14, Lemma 2.2] (or checked directly) yields $\mathsf{m}(1\,;g_0)u \in \mathsf{H}(\mathbb{Q})$ $(\exists u \in \mathsf{N}(\mathbb{Q}))$, which gives the second equality of the following formula.

$$\begin{aligned}\mathsf{H}(\mathbb{Q})\mathsf{m}(t\,;g_0)\mathsf{N}(\mathbb{Q}) &= \mathsf{H}(\mathbb{Q})(\mathsf{m}(1\,;g_0)\,u)\cdot\mathsf{m}(t\,;\mathtt{I}_q)\cdot(\mathsf{m}(t\,;\mathtt{I}_q)^{-1}u^{-1}\,\mathsf{m}(t\,;\mathtt{I}_q))\mathsf{N}(\mathbb{Q})\\ &= \mathsf{H}(\mathbb{Q})\mathsf{m}(t\,;\mathtt{I}_q)\mathsf{N}(\mathbb{Q})\,.\end{aligned}$$

This shows $\mathsf{H}(\mathbb{Q})\mathsf{P}(\mathbb{Q})$ is a union of cosets $\mathsf{H}(\mathbb{Q})\mathsf{m}(t\,;\mathtt{I}_q)\mathsf{N}(\mathbb{Q})$ $(t \in E^\times)$. Their disjointness is easily proved by [14, Lemma 2.2]. This proves the first assertion in (1).

(2) It is obvious that $\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{P}(\mathbb{Q})$ is a union of cosets $\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{m}(t\,;g_0)\mathsf{N}(\mathbb{Q})$. Let $(t, g_0) \in E^\times \times \mathsf{G}_0(\mathbb{Q})$. Let $g_0 = g_0'' g_0'$ with $g_0' \in \mathfrak{X}_\mathbb{Q}$ and $g_0'' \in \mathsf{G}_{0,x_0}(\mathbb{Q})$. By Lemma 9, $\mathsf{m}(t\,;g_0'')u \in \Upsilon_0\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}$ $(\exists u \in \mathsf{N}(\mathbb{Q}))$. We have

$$\begin{aligned}&\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{m}(t\,;g_0)\mathsf{N}(\mathbb{Q})\\ &\quad= \mathsf{H}(\mathbb{Q})(\Upsilon_0^{-1}\mathsf{m}(t\,;g_0'')u\Upsilon_0)\cdot\Upsilon_0^{-1}\mathsf{m}(1\,;g_0')\cdot(\mathsf{m}(1\,;g_0')^{-1}u^{-1}\mathsf{m}(1\,;g_0'))\mathsf{N}(\mathbb{Q})\\ &\quad= \mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{m}(1\,;g_0')\mathsf{N}(\mathbb{Q})\,.\end{aligned}$$

Hence $\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{P}(\mathbb{Q})$ is a union of those cosets $\mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{m}(1\,;g_0')\mathsf{N}(\mathbb{Q})$ $(g_0' \in \mathfrak{X}_\mathbb{Q})$. Their disjointness follows from Lemma 9. This completes the proof. □

PROPOSITION 4. (1) *If* $\gamma \in \mathsf{H}(\mathbb{Q})\mathsf{m}(t\,;\mathtt{I}_q)\mathsf{N}(\mathbb{Q})$ *with* $t \in E^\times$, *then* $\mathsf{N}_\gamma = \{1\}$.

(2) *If* $\gamma \in \mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{m}(1\,;g_0)\mathsf{N}(\mathbb{Q})$ *with* $g_0 \in \mathfrak{X}_\mathbb{Q}$, *then*

$$\mathsf{N}_\gamma = \{\mathsf{n}(X\,;z) \in \mathsf{N}\,|\, S_0(X, g_0^{-1}x_0) = 0\}\,.$$

Proof. (1) Let $\gamma \in \mathsf{H}(\mathbb{Q})p$ with $p \in \mathsf{P}(\mathbb{Q})$. Since $\mathsf{H} \cap \mathsf{N} = \{1\}$ by [14, Lemma 2.2], $\mathsf{N}_\gamma = p^{-1}(\mathsf{H} \cap \mathsf{N})p = \{1\}$.

(2) Write $\gamma = h\Upsilon_0^{-1}p$ with $h \in \mathsf{H}(\mathbb{Q})$, $p = \mathsf{m}(1\,;g_0)\mathsf{n}(Y\,;w)$ and $g_0 \in \mathfrak{X}_\mathbb{Q}$. Then $\mathsf{N}_\gamma = p^{-1}(\Upsilon_0\mathsf{H}\Upsilon_0^{-1} \cap \mathsf{N})p$. Then by Lemma 9 we have the conclusion. □

PROPOSITION 5. (1) *For* $\gamma = \mathsf{H}(\mathbb{Q})\mathsf{m}(n\,;\mathtt{I}_q)\mathsf{N}(\mathbb{Q})$ *with* $n \in E^\times$, *we have*

$$I_s(\gamma\,;g) = \int_{\mathsf{N}(\mathbb{A})} \phi_s(\mathsf{m}(n\,;\mathtt{I}_q)ug)\,du\,.$$

(2) *For* $\gamma = \mathsf{H}(\mathbb{Q})\Upsilon_0^{-1}\mathsf{m}(1\,;g_0)\mathsf{N}(\mathbb{Q})$ *with* $g_0 \in \mathfrak{X}_\mathbb{Q}$, *we have*

$$I_s(\gamma\,;g) = \int_{E_\mathbb{A}} \phi_s(\Upsilon_0^{-1}\mathsf{m}(1\,;g_0)\mathsf{n}(g_0^{-1}x_0\lambda\,;-2^{-1}\mathrm{N}(\lambda)\Delta)g)\,d\lambda\,.$$

Proof. These claims follow from Lemma 8 and Proposition 4. To show (2) we should note that the map $\lambda(\in E_\mathbb{A}) \mapsto \mathsf{N}_\gamma(\mathbb{A})\mathsf{n}(g_0^{-1}x_0\lambda\,;-2^{-1}\mathrm{N}(\lambda)\Delta)$ is a measure-preserving bijection from $E_\mathbb{A}$ onto $\mathsf{N}_\gamma(\mathbb{A})\backslash\mathsf{N}(\mathbb{A})$. □

Lemma 8, Proposition 3 and Proposotion 5 give the following expression of the constant term.

PROPOSITION 6. *We have* $F_{s,(\mathsf{P})}(g)=\Phi_s(g)+\varphi_s(g)$ *with* $\Phi_s(g)=\sum_{n\in E^\times}\Phi_s(n\,;g)$ *and* $\varphi_s(g)=\sum_{g_0\in\mathfrak{X}_\mathbb{Q}}\varphi_s(g_0\,;g)$, *where*

$$\Phi_s(n\,;g):=\Phi_s^{(\infty)}(n\,;g_\infty)\prod_{p\in I(\mathbb{Q})}\Phi^{(p)}(n\,;g_p)$$

with

$$\Phi_s^{(\infty)}(n\,;g_\infty):=\int_{\mathsf{N}(\mathbb{R})}\phi_s^{(\infty)}(\mathsf{m}(n\,;\mathtt{I}_q)u_\infty g_\infty)du_\infty\,,$$

$$\Phi^{(p)}(n\,;g_p):=\int_{\mathsf{N}(\mathbb{Q}_p)}\phi^{(p)}(\mathsf{m}(n\,;\mathtt{I}_q)u_pg_p)du_p\,,$$

and

$$\varphi_s(z_0;g):=\varphi_s^{(\infty)}(g_0\,;g_\infty)\prod_{p\in I(\mathbb{Q})}\varphi^{(p)}(g_0\,;g_p)$$

with

$$\varphi_s^{(\infty)}(g_0\,;g_\infty):=\int_{E_\mathbb{R}}\phi_s^{(\infty)}(\Upsilon_0^{-1}\mathsf{m}(1\,;g_0)\mathsf{n}(g_0^{-1}x_0\lambda_\infty\,;-2^{-1}\mathrm{N}(\lambda_\infty)\Delta)g_\infty)d\lambda_\infty\,,$$

$$\varphi^{(p)}(g_0\,;g_p):=\int_{E_p}\phi^{(p)}(\Upsilon_0^{-1}\mathsf{m}(1\,;g_0)\mathsf{n}(g_0^{-1}x_0\lambda_p\,;-2^{-1}\mathrm{N}(\lambda_p)\Delta)g_p)d\lambda_p\,.$$

In the next two subsections, we shall study the local integrals appearing in the above formula.

5.2. Local integrals at Archimedean place

We calculate the integrals $\Phi_s^{(\infty)}(n\,;g_\infty)$ and $\varphi_s^{(\infty)}(g_0\,;g_\infty)$. We first prove a lemma.

LEMMA 10. *Let* $g=\mathsf{m}(t\,;g_0)\mathsf{n}(X\,;z)\in\mathsf{P}(\mathbb{R})$ *with* $t>0$, $g_0\in\mathsf{G}_0(\mathbb{R})$ *and* $(X,z)\in E_\mathbb{R}^q\times E_\mathbb{R}$, $z=-2^{-1}S_0[X]+y$ $(y\in E_\mathbb{R}^{(0)})$.

(1) *We have* $g\in\mathsf{H}(\mathbb{R})\,a_rK_\infty$ *with* $r\in\mathbb{R}$ *such that*

$$(\cosh r)^2=1+\frac{1}{2\Delta}\left(t-\frac{\Delta}{2t}+\frac{t}{2}S_0[g_0X-t^{-1}\mathsf{a}]\right)^2+\frac{1}{2\Delta}\left\|\frac{S_0(g_0X,\mathsf{a})-S_0(\mathsf{a},g_0X)}{2}+ty\right\|^2.$$

(2) *We have* $\Upsilon_0^{-1}g\in\mathsf{H}(\mathbb{R})\,a_rK_\infty$ *with* $r\in\mathbb{R}$ *such that*

$$(\cosh r)^2=1+\frac{1}{2\Delta}\left\|\frac{1}{t}+S_0(g_0X,x_0)\right\|^2.$$

Proof. By the decomposition $\mathsf{G}(\mathbb{R})=\mathsf{H}(\mathbb{R})A_\infty K_\infty$, we can write $g=ha_rk$ with some $h\in\mathsf{H}(\mathbb{R})$, $k\in K_\infty$ and $r\geqslant 0$. Hence $\|g^{-1}\mathsf{v}_\eta^+\|_S^2=\|a_r^{-1}\mathsf{v}_\eta^+\|_S^2=2\cosh^2 r-1$ on one hand. On the other hand, since $g^{-1}\mathsf{v}_\eta^+=\Delta^{-1/2}\begin{bmatrix}t^{-1}+S_0(\mathsf{a},X)+z^\iota t^\iota\\ \mathsf{a}-Xt^\iota\\ t^\iota\end{bmatrix}$, we have another expression of $\|g^{-1}\mathsf{v}_\eta^+\|_S^2$. Comparing these two expressions, we obtain the formula in (1). (2) is proved similarly. □

5.2.1. *The functions $\Phi_s^{(\infty)}$.* We first consider the functions $\Phi_s^{(\infty)}(n\,;\,g)$ with $n \in E^\times$.

PROPOSITION 7. *Let $g = \mathsf{m}(t\,;\,m_0)$ with $(t, m_0) \in E_\mathbb{R}^\times \times \mathsf{G}_0(\mathbb{R})$. Then we have*

$$\Phi_s^{(\infty)}(n\,;\,g) = \gamma(s)\,\{1+\beta_\Delta(\|nt\|)\}^{\delta/2}\{1-\beta_\Delta(\|nt\|)\}^{(s-\rho_0)/2}$$
$$\times\, {}_2F_1\left(\frac{-s+\rho_0-\delta+2}{2}, \frac{s-\rho_0+\delta}{2}\,;\,\frac{s+\rho_0-\delta+2}{2}\,;\,\frac{1-\beta_\Delta(\|nt\|)}{2}\right),$$

where

$$\beta_\Delta(r) := \frac{1-2^{-1}\Delta r^{-2}}{1+2^{-1}\Delta r^{-2}}, \quad r \in \mathbb{R},$$
$$\gamma(s) := |\det(Q_0)|^{-1/2}2^{(-s+\rho_0)/2+\delta q}\pi^{\rho_0-\delta/2+1}D_\mathscr{O}^{-(q+1)/2}\Delta^{\rho_0}\|n\|^{-2\rho_0}$$
$$\times\,\frac{\Gamma(s)}{\Gamma((s+\rho_0)/2)\Gamma((s+\rho_0-\delta+2)/2)}.$$

In particular $\Phi^{(\infty)}(n\,;\,g)$ is independent of m_0.

Proof. Noting $\mathsf{m}(n\|n\|^{-1}\,;\,\mathtt{I}_q)$, $\mathsf{m}(1; m_0) \in K_\infty$ and $\phi_s^{(\infty)}$ is right K_∞-invarinat, we have

(5.3)

$$\begin{aligned}
&\Phi_s^{(\infty)}(n\,;\,g)\\
&= \int_{E_\mathbb{R}^q}\int_{E_\mathbb{R}^{(0)}} \phi_s^{(\infty)}(\mathsf{m}(\|n\|t\,;\,\mathtt{I}_q)\,\mathsf{n}(t^{-1}\|n\|^{-1}Xn^\iota\,;\,t^{-2}\|n\|^{-2}n(y-2^{-1}S_0[X])n^\iota))\,dXdy\\
&= \int_{E_\mathbb{R}^q}\int_{E_\mathbb{R}^{(0)}} \phi_s^{(\infty)}(\mathsf{m}(\|n\|t\,;\,\mathtt{I}_q)\,\mathsf{n}(t^{-1}X'\,;\,t^{-2}(y-2^{-1}S_0[X'])))\,dX'dy'.
\end{aligned}$$

Here to obtain the second equality we made a measure-preserving change of variables by $X' = X\|n\|(n^\iota)^{-1}$, $y' = \|n\|^2n^{-1}y(n^\iota)^{-1}$. By Lemma 10, the integrand in the last integral becomes

$$\left\{1+\frac{1}{2\Delta}\left(\|n\|t-\frac{\Delta}{2\|n\|t}+\frac{\|n\|t}{2}S_0[Z]\right)^2+\frac{1}{2\Delta}\|x\|^2\right\}^{-(s+\rho_0)/2}$$

with

$$Z := t^{-1}(X'-\|n\|^{-1}\mathsf{a}), \quad x := t^{-1}\{\|n\|y'-2^{-1}(S_0(X,\mathsf{a})-S_0(\mathsf{a},X))\}.$$

Hence from the last formula of (5.3), the integral $\Phi_s^{(\infty)}(n\,;\,g)$ equals

$$t^{\delta q+\delta-1}\|n\|^{1-\delta}\int_{E_\mathbb{R}^q}dZ\int_{E_\mathbb{R}^{(0)}}dx\left\{1+\frac{1}{2\Delta}\left(\|n\|t-\frac{\Delta}{2\|n\|t}+\frac{\|n\|t}{2}S_0[Z]\right)^2+\frac{1}{2\Delta}\|x\|^2\right\}^{-(s+\rho_0)/2}.$$

We change variable again by $x' := (2\Delta)^{1/2}\{1+\frac{1}{2\Delta}(\|n\|t-\frac{\Delta}{2\|n\|t}+\frac{\|n\|t}{2}S_0[Z])^2\}^{1/2}x$ to see that the integral $\Phi^{(\infty)}(n\,;\,g)$ is $(2\Delta)^{\delta-1}t^{\delta-1}\|n\|^{1-\delta}$ times the product of the two integrals:

- $J_1 := t^{\delta q}\displaystyle\int_{E_{\mathbb{R}}^{q}}\left\{1+\frac{1}{2\Delta}\left(\|n\|t-\frac{\Delta}{2\|n\|t}+\frac{\|n\|t}{2}S_0[Z]\right)^2\right\}^{-(s+\rho_0-\delta+1)/2}dZ;$
- $J_2 := \displaystyle\int_{E_{\mathbb{R}}^{(0)}}(1+\|x'\|^2)^{-(s+\rho_0)/2}\,dx'.$

By adopting the polar coordinate on $E_{\mathbb{R}}^{(0)} \cong \mathbb{R}^{\delta-1}$ and making a suitable change of variables, J_2 is transformed to a beta-integral, which is easily evaluated as $J_2 = 2^{\delta/2-1}D_{\mathscr{O}}^{-1/2}\pi^{(\delta-1)/2}\Gamma((s+\rho_0-\delta+1)/2)\Gamma((s+\rho_0)/2)^{-1}$. Let us the integral J_1. Put $m := 2^{-1}\delta q = \rho_0-\delta+1$. By an orthonormal basis we identify $E_{\mathbb{R}}^{q}$ with $\mathbb{R}^{2m}$. Noting the Jacobian factor $\det(2^{-1}Q_0)^{-1/2}$ produced by this identification, we introduce the polar coordinate on $\mathbb{R}^{2m}$ to obtain

$$\begin{aligned}J_1 &= t^{2m}\det(2^{-1}Q_0)^{-1/2}(2^{\delta/2}D_{\mathscr{O}}^{-1/2})^q A^{(2m-1)}\\ &\quad\times\int_0^\infty r^{2m-1}\left\{T^2\left(\frac{r}{2}\right)^4+(T^2-1)\left(\frac{r}{2}\right)^2+\frac14\left(T+\frac1T\right)^2\right\}^{-(s+m)/2}dr\end{aligned}$$

with $T := 2^{1/2}\Delta^{-1/2}\|n\|t$ and $A^{(2m-1)} = 2\pi^m\Gamma(m)^{-1}$. Let us change variable from r to u by $r=\sqrt{2u}(1+T^{-2})^{1/2}$. Then after a computation we arrive at the formula expressing J_1

$$\begin{aligned}&t^{2m}2^{2m-1+\delta q}\det(Q_0)^{-1/2}D_{\mathscr{O}}^{-q/2}A^{(2m-1)}T^{-m}\left(\frac{T+T^{-1}}{2}\right)^{-s}\\ &\quad\times\int_0^\infty u^{m-1}(u^2+2\beta u+1)^{-(s+m)/2}\,du\end{aligned}$$

with $\beta := (T^2-1)(T^2+1)^{-1}$. Let us utilize the formula [7, 3.252,10 (p. 297)] to evaluate the last integral. Consequently we have

$$\begin{aligned}J_1 &= 2^{(-s+m+1)/2+\delta q}\pi^{m+1/2}D_{\mathscr{O}}^{-q/2}\Delta^m\|n\|^{-2m}\det(Q_0)^{-1/2}\Gamma(s)\Gamma((s+m)/2)^{-1}\\ &\quad\times(1+\beta)^m(1-\beta)^{(s-3m+1)/4}P_{(s-m-1)/2}^{(-s-m+1)/2}(\beta)\end{aligned}$$

with $P_\nu^\mu(\beta)$ the Legendre function, which is given as

$$P_\nu^\mu(\beta) = \Gamma(1-\mu)^{-1}(1+\beta)^{\mu/2}(1-\beta)^{-\mu/2}{}_2F_1(-\nu,1+\nu\,;1-\mu\,;2^{-1}(1-\beta))\,,\qquad \beta\in\mathbb{R},\ |\beta|<1\,.$$

The remaining part of the proof is just an elementary calculation, which can be omitted. □

The estimate of $\Phi_s^{(\infty)}(n\,;\mathsf{m}(t\,;\mathtt{I}_q))$ for large $t>0$ is given as follows.

PROPOSITION 8. *Given a large number R and a compact set $B\subset\{s\in\mathbb{C}|\operatorname{Re}(s)>\rho_0\}$, we can take a positive constant $C_{R,B}$ such that*

$$|\Phi_s^{(\infty)}(n\,;\mathsf{m}(t\,;\mathtt{I}_q))| \leqslant C_{R,B}\,\|n\|^{-\operatorname{Re}(s)-\rho_0}t^{-\operatorname{Re}(s)+\rho_0},\quad \forall t>R,\ \forall s\in B,\ \forall n\in E^\times\,.$$

Proof. Note that $x = 2^{-1}(1 - \beta_\Delta(r))$ lies in the interval [0, 1] and

$$\sup_{(s,x)\in B\times[0,1]} {}_2F_1((-s + \rho_0 - \delta + 2)/2, (s - \rho_0 + \delta)/2\,; (s + \rho_0 - \delta + 2)/2\,; x) < +\infty\,.$$

Hence we have a constant C'_B such that

$$|\Phi_s^{(\infty)}(n\,; \mathsf{m}(t\,; \mathtt{I}_q))| \leqslant C'_B\|n\|^{-2\rho_0}\{1 + \beta_\Delta(\|n\|t)\}^{\delta/2}\{1 - \beta_\Delta(\|n\|t)\}^{(\mathrm{Re}(s)-\rho_0)/2}\,, \quad \forall s \in B,\ \forall t > 0\,.$$

From this, we obtain the estimation of the required form readily. □

For later use we compute the Mellin transform of $\Phi_s^{(\infty)}(n\,; \mathsf{m}(t\,; \mathtt{I}_q))$.

PROPOSITION 9. *For* $(s, \nu) \in \mathbb{C}^2$ *with* $\mathrm{Re}(s) > \mathrm{Re}(\nu) > \rho_0 - \delta$, *the integral*

$$\widehat{\Phi}_s^{(\infty)}(n\,; \nu) := \int_0^\infty \Phi_s^{(\infty)}(s\,; \mathsf{m}(t\,; \mathtt{I}_q))t^{\nu-\rho_0}\frac{dt}{t}$$

converges absolutely and locally uniformly with respec to (s, ν) *and equals* $\gamma_{S_0,\Delta}(s, \nu)\|n\|^{-\nu-\rho_0}$ *with*

$$\gamma_{S_0,\Delta}(s, \nu) :=|\det(Q_0)|^{-1/2}2^{(\rho_0-\nu+\delta)/2+\delta q-1}\pi^{\rho_0-\delta/2+1}D_{\mathscr{O}}^{-(q+1)/2}\Delta^{(\nu+\rho_0)/2} \times \frac{\Gamma((s-\nu)/2)\Gamma((s+\nu)/2)\Gamma((\nu-\rho_0+\delta)/2)}{\Gamma((s+\rho_0)/2)\Gamma((s-\rho_0+\delta)/2)\Gamma((\nu+\rho_0-\delta+2)/2)}\,.$$

Proof. This follows from 7 by means of [7, 7.512,3 (p. 849)] after a change of variable $t = (2^{-1}\Delta)^{1/2}\|n\|^{-1}(x^{-1} - 1)^{1/2}$. □

5.2.2. *The functions* $\varphi_s^{(\infty)}$. We consider the functions $\varphi_s^{(\infty)}(g_0\,; g)$ with $g_0 \in \mathfrak{X}_{\mathbb{Q}}$.

PROPOSITION 10. *Let* $g = \mathsf{m}(t\,; m_0)$ *with* $(t, m_0) \in E_{\mathbb{R}}^\times \times \mathsf{G}_0(\mathbb{R})$. *Then we have*

$$\varphi_s^{(\infty)}(g_0\,; g) = (2\pi\Delta^{-1})^{\delta/2}\frac{\Gamma((s+\rho_0-\delta)/2)}{\Gamma((s+\rho_0)/2)}\|t\|^\delta\,.$$

In particular it is independent of m_0.

Proof. By Lemma 10 (2) and after a change of variables, the integral can be written by a beta-integral. □

5.3. Local integrals at non-Archimedean places

Let p be a prime. Before we calculate the integrals $\Phi_s^{(p)}(n\,; g_p)$ and $\varphi_s^{(\infty)}(g_0\,; g_p)$, we first prove a lemma, making one assumption on the p-adic norm of Δ:

$$|\Delta|_p = 1 \text{ if } E_p \text{ is not a division algebra}\,.$$

We also assume that $2 \notin R(E)$ if $\delta = 2$. We keep these assumptions throughout this subsection.

LEMMA 11. *Let* $g \in \mathsf{G}(\mathbb{Q}_p)$. *Then* $g \in \mathsf{H}(\mathbb{Q}_p)\,K_p$ *if and only if* $S(\mathscr{L}_p, g^{-1}\eta) = \mathscr{O}_p$.

The proof of this lemma in turn depends on the following auxiliary lemma.

LEMMA 12. *Let* $\mathsf{x} \in S^{-1}\mathscr{O}_p^{q+2}$ *be a vector such that* $S(\mathscr{L}_p, \mathsf{x}) = \mathscr{O}_p$. *We assume* $S[\mathsf{x}] \in \mathbb{Z}_p^\times$ *if* E_p *is not a division algebra. Then there exists a* $k \in K_p$ *such that* $k\mathsf{x} = \begin{bmatrix} * \\ * \\ 1 \end{bmatrix}$.

First let us prove Lemma 11 using Lemma 12. We extend a part of the argument in the proof of [14, Proposition 3.9] to include the case when E_p is non-division. The implication that $g \in \mathsf{H}(\mathbb{Q}_p)K_p$ means $S(\mathscr{L}_p, g^{-1}\eta) = \mathscr{O}_p$ is obvious. We prove the converse. Assume $S(\mathscr{L}_p, \mathsf{x}) = \mathscr{O}_p$ with $\mathsf{x} = g^{-1}\eta$. By Lemma 12, we can find some $k_1 \in K_p$ such that $k_1\mathsf{x} = \begin{bmatrix} a' \\ \mathsf{a}' \\ 1 \end{bmatrix}$ with $a' \in E_p$, $\mathsf{a}' \in E_p^q$. Since k_1 preserves $S^{-1}\mathscr{O}_p^{q+2}$ we have $a' \in \mathscr{O}_p$ and $\mathsf{a}' \in S_0^{-1}\mathscr{O}_p^q$. When E_p is a division algebra, the obvious identity $S[\eta] = S[k_1\mathsf{x}]$ gives $S_0[\mathsf{a}] = S_0[\mathsf{a}']$. Hence by [14, Lemma 3.8] we have $\mathsf{b} := k_0\mathsf{a}' - \mathsf{a} \in \mathscr{O}_p^q$ with some $k_0 \in K_{0,p} := \{k_0 \in \mathsf{G}_0(\mathbb{Q}_p) \,|\, k_0\mathscr{O}_p^q = \mathscr{O}_p^q\}$. When E_p is not a division algebra, then $S_0\mathscr{O}_p^q = \mathscr{O}_p^q$ so that $\mathsf{x} \in \mathscr{O}_p^{q+2}$ and $\mathsf{a}' \in \mathscr{O}_p^q$. Hence in this case $\mathsf{b} := k_0\mathsf{a}' - \mathsf{a} \in \mathscr{O}_p^q$ with $k_0 = \mathtt{I}_q$. Choose $b \in \mathscr{O}_p$ such that $S_0[\mathsf{b}] + \tau(b) = 0$ and put $k_2 = \mathsf{n}(-\mathsf{b}\,; b)\mathsf{m}(1\,; k_0)$. Then $k_2 \in K_p$ and $k_2k_1\mathsf{x} = \begin{bmatrix} b' \\ \mathsf{a} \\ 1 \end{bmatrix}$ $(\exists b' \in \mathscr{O}_p)$. The identity $2 + S_0[\mathsf{a}] = S[\eta] = S[k_2k_1\mathsf{x}]$ gives $\tau(1-b') = 0$. Put $k_3 = \mathsf{n}(0\,; 1-b')$. Then $k_3 \in K_p$ and $k_3k_2k_1\mathsf{x} = \eta$, or equivalently $k_3k_2k_1g^{-1} \in \mathsf{H}(\mathbb{Q}_p)$. Therefore $g \in \mathsf{H}(\mathbb{Q}_p)K_p$ as desired.

Now we proceed to prove Lemma 12. When E_p is division, the statement is proved in [14, Lemma 3.4]. Assume E_p is not division. Then we have two possibilities: (a) $E_p \cong \mathbb{Q}_p \oplus \mathbb{Q}_p$, (b) $E_p \cong \mathrm{M}_2(\mathbb{Q}_p)$.

- We first treat the case (a). The involution ι on $E_p = \mathbb{Q}_p \oplus \mathbb{Q}_p$ is given by $(x', x'')^\iota = (x'', x')$ and the integer ring is $\mathscr{O}_p = \mathbb{Z}_p \oplus \mathbb{Z}_p$. We identify $\mathrm{M}_{lm}(E_p) = \mathrm{M}_{lm}(\mathbb{Q}_p) \oplus \mathrm{M}_{lm}(\mathbb{Q}_p)$ and similarly for $\mathrm{M}_{lm}(\mathscr{O}_p)$. Then $S = (T, {}^tT)$ with $T = \begin{bmatrix} & & 1 \\ & T_0 & \\ 1 & & \end{bmatrix}$ for some $T_0 \in \mathrm{GL}_q(\mathbb{Q}_p)$. For $g = (g', g'') \in \mathrm{M}_{q+2}(E_p)$, the condition $g \in \mathsf{G}(\mathbb{Q}_p)$ is equivalent to $g'' = {}^tT^{-1}\,{}^t(g')^{-1}\,{}^tT$. Hence the first projection $\mathrm{M}_{q+2}(E_p) \to \mathrm{M}_{q+2}(\mathbb{Q}_p)$ gives an identification $\mathsf{G}(\mathbb{Q}_p) = \mathrm{GL}_{q+2}(\mathbb{Q}_p)$, $K_p = \mathrm{GL}_{q+2}(\mathbb{Z}_p)$. Moreover the natural action of $g \in \mathsf{G}(\mathbb{Q}_p) = \mathrm{GL}_{q+2}(\mathbb{Q}_p)$ on E_p^{q+2} is given by
$$g(X', X'') = (gX', {}^tT^{-1}\,{}^tg^{-1}\,{}^tTX''), \quad (X', X'') \in E_p^{q+2}.$$
Write $\mathsf{x} = (\mathsf{x}', \mathsf{x}'')$ with $\mathsf{x}', \mathsf{x}'' \in \mathbb{Q}_p^{q+2}$. Then by the assumption $S(\mathscr{L}_p, \mathsf{x}) = \mathscr{O}_p$, the two vectors $\mathsf{x}', \mathsf{x}''$ are primitive in $\mathbb{Z}_p^{q+2}$. Hence there exists a matrix $k_1 \in K_p = \mathrm{GL}_{q+2}(\mathbb{Z}_p)$ such that $k_1\mathsf{x} = \left(\begin{bmatrix} 0 \\ \mathsf{o}_q \\ 1 \end{bmatrix}, \mathsf{y}\right)$ with $\mathsf{y} := {}^tT^{-1}\,{}^tk_1^{-1}\,{}^tT\mathsf{x}''$. Note that y is still primitive in $\mathbb{Z}_p^{q+2}$. Write $\mathsf{y} = \begin{bmatrix} y_1 \\ \mathsf{y}_2 \\ y_3 \end{bmatrix}$ with $\mathsf{y}_2 \in \mathbb{Z}_p^q$, $y_1, y_3 \in \mathbb{Z}_p$. We first consider the case when $y_3 \in \mathbb{Z}_p^\times$. Put $k_2 := \mathrm{diag}(y_3, \mathtt{I}_{q+1}) \in K_p$. Then $k_2k_1\mathsf{x} = \left(\begin{bmatrix} 0 \\ \mathsf{o}_q \\ 1 \end{bmatrix}, \begin{bmatrix} y_1 \\ \mathsf{y}_2 \\ 1 \end{bmatrix}\right)$. This complete the proof. We consider the case when $y_3 \notin \mathbb{Z}_p^\times$. Then $\begin{bmatrix} y_1 \\ \mathsf{y}_2 \end{bmatrix}$ should be primitive in $\mathbb{Z}_p^{q+1}$. Therefore we can find some $A_{12} \in \mathrm{M}_{1q}(\mathbb{Z}_p)$ and $a_{13} \in \mathbb{Z}_p$ such that

$A_{12}{}^tT_0\mathsf{y}_2 + a_{13}y_1 = 1 - y_3$. Put $k_3 := {}^t\begin{bmatrix} 1 & A_{12} & a_{13} \\ & 1_q & \\ & & 1 \end{bmatrix}^{-1} \in K_p$. Then a computation shows that $k_3k_1\mathsf{x} = \left(\begin{bmatrix} * \\ * \\ 1 \end{bmatrix}, \begin{bmatrix} * \\ * \\ 1 \end{bmatrix}\right)$. This completes the proof.

- We treat the case (b). We may assume that the involution of $E_p = \mathrm{M}_2(\mathbb{Q}_p)$ is given by $a^\iota = J_2^{-1}\,{}^ta\,J_2$ with $J_l := \begin{bmatrix} 0 & -1_l \\ 1_l & 0 \end{bmatrix}$ for $l \in \mathbb{N}$, and that $\mathscr{O}_p = \mathrm{M}_2(\mathbb{Z}_p)$. For positive inegers l and m, let us identify the space $\mathrm{M}_{lm}(E_p)$ with $\mathrm{M}_{2l,2m}(\mathbb{Q}_p)$ by the map $g \mapsto \tilde{g}$ defined as follows. Given $g = (g_{ij})_{1\leqslant i\leqslant l, 1\leqslant j\leqslant m} \in \mathrm{M}_{l,m}(E_p)$, we write its (i,j)-th entry as $g_{ij} = (g_{ij}^{\alpha\beta})_{1\leqslant \alpha,\beta\leqslant 2} \in \mathrm{M}_2(\mathbb{Q}_p)$ and put $g^{(\alpha\beta)} := (g_{ij}^{\alpha\beta})_{1\leqslant i\leqslant l, 1\leqslant j\leqslant m} \in \mathrm{M}_{l,m}(\mathbb{Q}_p)$. Then set $\tilde{g} := \begin{bmatrix} g^{(11)} & g^{(12)} \\ g^{(21)} & g^{(22)} \end{bmatrix}$. Obviously this identification brings $\mathrm{M}_{lm}(\mathscr{O}_p)$ to $\mathrm{M}_{2l,2m}(\mathbb{Z}_p)$. Moreover $(gX)^\sim = \tilde{g}\tilde{X}$ for two matrices with coefficients in E_p and $X^{*\sim} = J_l{}^t\tilde{X}J_l$ for any $X \in \mathrm{M}_l(E_p)$.

  Now put $T_0 := J_q\tilde{S}_0$ and $T := J_{q+2}\tilde{T} \in \mathrm{M}_{2(q+2)}(\mathbb{Q}_p)$. Then T_0 and T are unimodular symplectic matrices by maximality of S_0. By the map $g \mapsto \tilde{g}$ the group $\mathsf{G}(\mathbb{Q}_p)$ is identified with $\mathrm{Sp}(T) := \{\tilde{g} \in \mathrm{GL}_{2(q+2)}(\mathbb{Q}_p) \mid {}^t\tilde{g}T\tilde{g} = T\}$ and K_p with $\mathrm{Sp}(T)_{\mathbb{Z}_p} := \mathrm{Sp}(T) \cap \mathrm{GL}_{2(q+2)}(\mathbb{Z}_p)$. Moreover for $X \in E_p^{q+2}$ with $\tilde{X} = (X', X'')$, $X', X'' \in \mathbb{Q}_p^{2(q+2)}$, we have $S[X] = {}^tX'TX''$. From this and the assumption $S(\mathscr{L}_p, \mathsf{x}) = \mathscr{O}_p$, $\Delta = S[\mathsf{x}] \in \mathbb{Z}_p^\times$, if we write $\tilde{\mathsf{x}} = (\mathsf{x}', \mathsf{x}'')$ with $\mathsf{x}', \mathsf{x}'' \in \mathbb{Q}_p^{2(q+2)}$, the vectors x' and x'' should be primitive in $\mathbb{Z}_p^{2(q+2)}$. Let us consider the four vectors $\mathsf{u} = \begin{bmatrix} 1 \\ 0_q \\ 0 \\ 0 \\ 0_q \\ 0 \end{bmatrix}$, $\mathsf{u}' = \begin{bmatrix} 0 \\ 0_q \\ 0 \\ 0 \\ 0_q \\ 1 \end{bmatrix}$, $\mathsf{w} = \begin{bmatrix} 0 \\ 0_q \\ 1 \\ 0 \\ 0_q \\ 0 \end{bmatrix}$, $\mathsf{w}' = \begin{bmatrix} 0 \\ 0_q \\ 0 \\ 1 \\ 0_q \\ 0 \end{bmatrix}$ in $\mathbb{Z}_p^{2(q+1)}$, Then ${}^t\mathsf{u}'T\mathsf{u} = {}^t\mathsf{w}'T\mathsf{w} = 1$ and ${}^t\mathsf{w}T\mathsf{u} = {}^t\mathsf{w}'T\mathsf{u} = {}^t\mathsf{w}T\mathsf{u}' = {}^t\mathsf{w}'T\mathsf{u}' = 0$. Hence we have ${}^t\mathsf{u}'T(\Delta\mathsf{u} + \mathsf{w}) = {}^t\mathsf{x}''T\mathsf{x}' = \Delta \in \mathbb{Z}_p^\times$, which means both $\mathbb{Z}_p$-modules $\mathbb{Z}_p\mathsf{x}'' + \mathbb{Z}_p\mathsf{x}'$ and $\mathbb{Z}_p\mathsf{u}' + \mathbb{Z}_p(\Delta\mathsf{u} + \mathsf{w})$ have orthogonal complements in $\mathbb{Z}_p^{2(q+2)}$ with respect to T. Moreover we can find some $k \in \mathrm{Sp}(T)_{\mathbb{Z}_p}$ such that $k(\mathsf{x}') = \Delta\mathsf{u} + \mathsf{w}$ and $k(\mathsf{x}'') = \mathsf{u}'$. Hence $k\tilde{\mathsf{x}} = \left(\begin{bmatrix} * \\ * \\ 1 \\ * \\ * \\ 0 \end{bmatrix}, \begin{bmatrix} * \\ * \\ 0 \\ * \\ * \\ 1 \end{bmatrix}\right)$.

  This completes the proof.

5.3.1. *The functions $\Phi^{(p)}$.* We consider the functions $\Phi^{(p)}(n\,;\,g)$ with $n \in E^\times$.

PROPOSITION 11. *For $t \in E_p^\times$, let $D^{(p)}(t)$ be the set of those elements $(X, y) \in E_p^q \times E_p^{(0)}$ such that*

$$\mathscr{O}_p\,(2t)^{-1}(S_0[X] - \Delta + y) + S_0(X, \mathscr{O}_p^q) + \mathscr{O}_p\,t^\iota = \mathscr{O}_p\,. \tag{5.4}$$

Then for $g = \mathsf{m}(t\,;\,m_0)$ with $(t, m_0) \in E_p^\times \times \mathsf{G}_0(\mathbb{Q}_p)$, we have

$$\Phi^{(p)}(n\,;\,g) = |2|_p^{1-\delta}|\mathrm{N}(n)|_p^{-\rho_0}|\det(m_0|E_p^q)|_p^{-1}\mathrm{mes}(D^{(p)}(nt))\,.$$

Proof. Let $(X, y) \in E_p^q \times E_p^{(0)}$. Then by a computation, we have

$$(\mathsf{m}(n\,;\,\mathtt{I}_q)\mathsf{n}(X\,;\,y - 2^{-1}S_0[X])g)^{-1} \cdot \eta = \begin{bmatrix} -2^{-1}(nt)^{-1}(S_0[Z] - \Delta + y') \\ Z \\ (nt)^\iota \end{bmatrix}$$

with $Z := m_0^{-1}(\mathsf{a} - Xn^\iota)\,(\in E_p^q)$, $-y' := nS_0(\mathsf{a}, X) - S_0(X, \mathsf{a})n^\iota - 2nyn^\iota\,(\in E_p^{(0)})$. Hence by Lemma 11, $\phi^{(p)}(\mathsf{m}(n\,;\,\mathtt{I}_q)\mathsf{n}(X\,;\,y - 2^{-1}S_0[X])g) = 1$ if and only if $(Z, y') \in D^{(p)}(nt)$. Since $dZ\,dy' = |2|_p^{\delta-1}|\mathrm{N}(n)|_p^{\rho_0}|\det(m_0|E_p^q)|_p^{-1}\,dX\,dy$, we obtain $\Phi^{(p)}(n\,;\,g) = |2|_p^{1-\delta}|\mathrm{N}(n)|_p^{-\rho_0}\mathrm{mes}(D^{(p)}(nt))$ as desired. □

COROLLARY 1. *Let $g = \mathsf{m}(t\,;\,m_0)$ be as in Proposition* 11. *If $nt \in E_p - \mathscr{O}_p$, then $\Phi^{(p)}(n\,;\,g) = 0$.*

Proof. It is obvious that $(X, y) \in D^{(p)}(nt)$ means $\mathscr{O}_p(nt)^\iota \subset \mathscr{O}_p$ and then $nt \in \mathscr{O}_p$. In particular if $nt \in E_p - \mathscr{O}_p$ then $D^{(p)}(nt) = \varnothing$, and hence $\Phi^{(p)}(n\,;\,g) = 0$. This proves the assertion. □

DEFINITION 1. (1) For given $D \in \mathbb{Q}_p$ and $\mathscr{B} \subset \mathbb{Q}_p$, set

$$A_{S_0,D}^{(p)}(\mathscr{B}) := \{X \in S_0^{-1}\mathscr{O}_p^q \,|\, S_0[X] - D \in \mathscr{B}\}\,.$$

(2) For $D \in \mathbb{Q}_p$ and $t \in \mathbb{Q}_p^\times$, set

$$\Gamma_D^{(p)}(t) := \frac{\mathrm{mes}(A_D^{(p)}(\tau(t\mathscr{O}_p)))}{\mathrm{mes}(\tau(t\mathscr{O}_p))}\,.$$

Let us see that $\mathrm{mes}(D^{(p)}(t))$ is expressed in terms of simpler quantities $\Gamma_D^{(p)}(t)$. First consider the case when E_p is a division algebra.

PROPOSITION 12. *Let $p \notin S(E)$ and fix a prime element ϖ of $\mathscr{O}_p$. Let $t \in \mathscr{O}_p \cap E_p^\times$. Then*

(5.5)

$$\mathrm{mes}(D^{(p)}(t)) = |2|_p^{\delta-1}|\mathrm{N}(t)|_p^{\delta/2}(\Gamma_\Delta^{(p)}(t) - \varepsilon^{(p)}(t)|\mathrm{N}(\varpi)|_p^{\rho_0-\delta/2}\Gamma_{\mathrm{N}(\varpi)^{-1}\Delta}^{(p)}(t\varpi^{-1}))\,,$$

where $\varepsilon^{(p)}(t)$ is 1 *if $t \in \varpi\mathscr{O}_p$ and* 0 *otherwise.*

Proof. Since $\mathscr{O}_p$ has a unique maximal ideal $\varpi\mathscr{O}_p = \mathscr{O}_p\varpi$, the condition $\mathscr{O}_p a + \mathscr{O}_p b + \mathscr{O}_p c = \mathscr{O}_p$ is equivalent to saying that at least one of a, b and c belongs to the units $\mathscr{O}_p^\times = \mathscr{O}_p - \varpi\mathscr{O}_p$. From this remark we have $D^{(p)}(t) = D_0(t) - D_1(t)$ with $D_\varepsilon(t)$ ($\varepsilon = 0, 1$) the set of those $(X, y) \in E_p^q \times E_p^{(0)}$ such that

$$S_0[X] - \Delta + y \in 2t\varpi^\varepsilon\mathscr{O}_p\,, \quad S_0(X, \mathscr{O}_p^q) \subset \mathscr{O}\varpi^\varepsilon\,, \quad t^\iota \in \varpi^\varepsilon\mathscr{O}_p\,. \tag{5.6}$$

Since $t \in \mathscr{O}_p$, the last condition $t^\iota \in \varpi^\varepsilon\mathscr{O}_p$ is automatic if $\varepsilon = 0$ and means $t \notin \mathscr{O}_p^\times$ if $\varepsilon = 1$. In particular $D_1(t) = \varnothing$ if $t \in \mathscr{O}_p^\times$. Since $D_1(t) \subset D_0(t)$, we have $\mathrm{mes}(D^{(p)}(t)) = \mathrm{mes}(D_0(t)) - \mathrm{mes}(D_1(t))$. Let us compute $\mathrm{mes}(D_\varepsilon(t))$ for $t \in \varpi^\varepsilon\mathscr{O}_p$. We make a change of variables by $X = X'\varpi^\varepsilon$ to have $\mathrm{mes}(D_\varepsilon) = |\mathrm{N}(\varpi)|_p^{\varepsilon\delta q/2}\mathrm{mes}(D'_\varepsilon)$ with D'_ε the set of those $(X', y) \in E_p^q \times E_p^{(0)}$ satisfying $y \in E^{(0)} \cap (-\mathrm{N}(\varpi)^\varepsilon S_0[X'] + \Delta + 2t\varpi^\varepsilon \mathscr{O}_p)$, $S_0(X', \mathscr{O}_p^q) \subset \mathscr{O}_p$. We first perform the integral with respect to y using Lemma 1 to obtain

(5.7)

$$\mathrm{mes}(D'_\varepsilon) = \frac{\mathrm{mes}(2t\varpi^\varepsilon \mathscr{O}_p)}{\mathrm{mes}(\tau(2t\varpi^\varepsilon \mathscr{O}_p))}\mathrm{mes}(\{X' \in S_0^{-1}\mathscr{O}_p^q \,|\, \mathrm{N}(\varpi)^\varepsilon S_0[X'] - \Delta \in \tau(t\varpi^\varepsilon \mathscr{O}_p)\}) .$$

From this (5.5) follows readily. □

Let us consider the case when E_p is not a division algebra.

PROPOSITION 13. *Let $p \in S(E)$. Assume $|\Delta|_p = 1$. Then we have*

$$\mathrm{mes}(D^{(p)}(t)) = |2|_p^{\delta-1}|\mathrm{N}(t)|_p^{\delta/2}\Gamma_\Delta^{(p)}(t)\,, \quad (\forall t \in \mathscr{O}_p \cap E_p^\times)\,.$$

Proof. For $p \in S(E)$, we have two possibilities: (a) $E_p \cong \mathbb{Q}_p \oplus \mathbb{Q}_p$, (b) $E_p \cong \mathrm{M}_2(\mathbb{Q}_p)$. We freely use the notation in the proof of Lemma 12.

- We treat the case (a). Put $t = (t_1, t_2) \in E_p^\times$ with $t_i \in \mathbb{Z}_p$. By changing the roles of t_1 and t_2 if necessary, we may assume $t_1 \in t_2\mathbb{Z}_p$. Then $D^{(p)}(t)$ is the set of those $(X_1, X_2, y) \in \mathbb{Q}_p^q \oplus \mathbb{Q}_p^q \times \mathbb{Q}_p$ such that

$$(2t_1)^{-1}({}^tX_2T_0X_1 - \Delta + y)\mathbb{Z}_p + {}^t\mathbb{Z}_p^qT_0X_1 + t_2\mathbb{Z}_p = \mathbb{Z}_p\,,$$
$$(2t_2)^{-1}({}^tX_2T_0X_1 - \Delta - y)\mathbb{Z}_p + {}^t\mathbb{Z}_p^qT_0X_2 + t_1\mathbb{Z}_p = \mathbb{Z}_p\,.$$

  The condition $(X_1, X_2, y) \in D^{(p)}(t)$ in particular means $X_1,\, X_2 \in \mathbb{Z}_p^q$, since $T_0 \in \mathrm{GL}_q(\mathbb{Z}_p)$. Put $y_0 := (2t_1)^{-1}({}^tX_2T_0X_1 - \Delta + y)$. Then by the substitution $y \mapsto y_0$ the set $D^{(p)}(t)$ is mapped bijectively onto the set D' of all those $(X_1, X_2, y_0) \in \mathbb{Z}_p^q \oplus \mathbb{Z}_p^q \times \mathbb{Q}_p$ satisfying

$$y_0\mathbb{Z}_p + {}^t\mathbb{Z}_p^qT_0X_1 + t_2\mathbb{Z}_p = \mathbb{Z}_p\,,$$
$$t_2^{-1}(t_1y_0 + {}^tX_2T_0X_1 - \Delta)\mathbb{Z}_p + {}^t\mathbb{Z}_p^qT_0X_2 + t_1\mathbb{Z}_p = \mathbb{Z}_p\,.$$

  Hence $\mathrm{mes}(D^{(p)}(t)) = |2t_1|_p\mathrm{mes}(D')$. In order to study $\mathrm{mes}(D')$ let us introduce the auxiliary sets $D'_{\varepsilon,\sigma}$ ($\varepsilon,\, \sigma \in \{0, 1\}$) consisting of all those $(X_1, X_2, y_0) \in \mathbb{Z}_p^q \oplus \mathbb{Z}_p^q \times \mathbb{Q}_p$ such that

$$\begin{aligned} &y_0 \in p^\varepsilon\mathbb{Z}_p, \quad X_1 \in p^\varepsilon\mathbb{Z}_p^q\,, \quad t_2 \in p^\varepsilon\mathbb{Z}_p\,, \\ &t_1y_0 + {}^tX_2T_0X_1 - \Delta \in t_2p^\sigma\mathbb{Z}_p\,, \quad X_2 \in p^\sigma\mathbb{Z}_p\,, \quad t_1 \in p^\sigma\mathbb{Z}_p\,. \end{aligned} \tag{5.8}$$

  Then as in the proof of Proposition 12, we have $D' = (D'_{00} - D'_{01}) - (D'_{10} - D'_{11})$ and hence $\mathrm{mes}(D') = \mathrm{mes}(D'_{00}) - \mathrm{mes}(D'_{01}) - \mathrm{mes}(D'_{10}) + \mathrm{mes}(D_{11})$. If $(\varepsilon, \sigma) \neq (0, 0)$, then the conditions (5.8) give $\Delta \in p\mathbb{Z}_p$, contradictory to $|\Delta|_p = 1$. Hence $D'_{\varepsilon\sigma} = \varnothing$ unless $(\varepsilon, \sigma) = (0, 0)$. We examine the set D'_{00}. Since $t_1 \in t_2\mathbb{Z}_p$, the conditions (5.8) reduced to the only one condition for $(X_1, X_2, y_0) \in \mathbb{Z}_p^q \oplus \mathbb{Z}_p^q \times \mathbb{Z}_p$ that ${}^tX_2T_0X_1 - \Delta \in t_2\mathbb{Z}_p$. Since $S_0[X] = {}^tX_2T_0X_1$ and since $\tau(t\mathscr{O}_p) = t_2\mathbb{Z}_p$, we consequently have $\mathrm{mes}(D'_{00}) = \mathrm{mes}(A_{S_0,\Delta}^{(p)}(\tau(t\mathscr{O}_p)))$. This completes the proof for the case (a).
- We treat the case (b). Put $t^\iota = (t_{ij})_{1\leqslant i,j\leqslant 2} \in E_p$ with $t_{ij} \in \mathbb{Z}_p$. By the form of the condition (5.4), we may assume that $t_{11} = p^\alpha$, $t_{22} = p^\beta$, $t_{21} = 0$ and $t_{12} \in R_\beta$ with

$\alpha, \beta \in \mathbb{N}$ and R_β a complete set of representatives of $\mathbb{Z}_p/p^\beta\mathbb{Z}_p$. The set $D^{(p)}(t)$ is then the set of all those $(X_1, X_2, y_{11}, y_{12}, y_{21}) \in \mathbb{Q}_p^{2q} \oplus \mathbb{Q}_p^{2q} \times \mathbb{Q}_p^3$ such that

$$((2p^\beta)^{-1}({}^tT_0X_1 - \Delta + y_{11}) + y_{21}t_{12}(2p^{\alpha+\beta})^{-1})\mathbb{Z}_p + p^\beta(2p^{\alpha+\beta})^{-1}y_{21}\mathbb{Z}_p$$
$$+ {}^t\mathbb{Z}_p^{2q}\,T_0X_1 + p^\alpha\mathbb{Z}_p = \mathbb{Z}_p\,,$$
$$((2p^{\alpha+\beta})^{-1}({}^tX_2T_0X_1 - \Delta - y_{11})t_{12} + p^\alpha y_{12}(2p^{\alpha+\beta})^{-1})\mathbb{Z}_p$$
$$+ (2p^\alpha)^{-1}({}^tX_2T_0X_1 - \Delta - y_{11})\mathbb{Z}_p + {}^t\mathbb{Z}_p^{2q}\,T_0X_2 + t_{12}\mathbb{Z}_p + p^\beta\mathbb{Z}_p = \mathbb{Z}_p\,.$$

We consider the case $\beta \geqslant \alpha$. Put

$$x_{21} := p^\beta y_{21}(2p^{\alpha+\beta})^{-1}\,,$$
$$x_{11} := (2p^\beta)^{-1}(({}^tX_2T_0X_1 - \Delta + y_{11}) + 2t_{12}x_{21})\,,$$
$$z := p^{-\alpha}(({}^tX_2T_0X_1 - \Delta) + t_{12}x_{21} - p^\beta x_{11})\,,$$
$$x_{12} := p^\alpha y_{12}(2p^{\alpha+\beta})^{-1} + zp^{-\beta}t_{12}\,.$$

Then through the substitution $y_{ij} \mapsto x_{ij}$ the set $D^{(p)}(t)$ is mapped bijectively onto the set D' of all those $(X_1, X_2, x_{11}, x_{12}, x_{21}) \in \mathbb{Z}_p^{2q} \oplus \mathbb{Z}_p^{2q} \times \mathbb{Z}_p^3$ such that

$$x_{11}\mathbb{Z}_p + x_{21}\mathbb{Z}_p + {}^t\mathbb{Z}_p^{2q}T_0X_1 + p^\alpha\mathbb{Z}_p = \mathbb{Z}_p\,,$$
$$x_{12}\mathbb{Z}_p + z\mathbb{Z}_p + {}^t\mathbb{Z}_p^{2q}T_0X_2 + t_{12}\mathbb{Z}_p + p^\beta\mathbb{Z}_p = \mathbb{Z}_p\,.$$

Noting the Jacobian factor produced by the substitution, we have $\mathrm{mes}(D^{(p)}(t)) = |2|_p|p^\alpha|_p^{-1}|p^{\alpha+\beta}|_p^2\mathrm{mes}(D')$. Now for $\sigma, \varepsilon \in \{0, 1\}$ let $D'_{\varepsilon\sigma}$ be the subset of $\mathbb{Z}_p^{2q} \oplus \mathbb{Z}_p^{2q} \times \mathbb{Z}_p^3$ defined by the relations

$$\text{(5.9)} \qquad x_{11} \in p^\varepsilon\mathbb{Z}_p\,, \quad x_{21} \in p^\varepsilon\mathbb{Z}_p\,, \quad X_1 \in p^\varepsilon\mathbb{Z}_p^{2q}\,, \quad p^\alpha \in p^\varepsilon\mathbb{Z}_p\,,$$
$$x_{12} \in p^\sigma\mathbb{Z}_p\,, \quad z \in p^\sigma\mathbb{Z}_p\,, \quad X_2 \in p^\sigma\mathbb{Z}_p^{2q}\,, \quad t_{12} \in p^\sigma\mathbb{Z}_p\,, \quad p^\beta \in p^\sigma\mathbb{Z}_p\,.$$

Then similarly as in case (a), we have $\mathrm{mes}(D') = \mathrm{mes}(D'_{00}) - \mathrm{mes}(D'_{01}) - \mathrm{mes}(D'_{10}) + \mathrm{mes}(D'_{11})$. If $(\varepsilon, \sigma) \neq (0, 0)$, then the condition (5.9) gives $\Delta \in p\mathbb{Z}_p$, contradictory to $|\Delta|_p = 1$. Hence $D_{\varepsilon\sigma} = \varnothing$ unless $\varepsilon = \sigma = 0$. Let us examine the set D'_{00}. Assume first $|t_{12}|_p \leqslant p^{-\alpha}$. Then (5.9) reduces to the condition ${}^tX_2T_0X_1 - \Delta \in p^\alpha\mathbb{Z}_p$ for $(X_1, X_2, x_{11}, x_{12}, x_{21}) \in \mathbb{Z}_p^{2q} \oplus \mathbb{Z}_p^{2q} \times \mathbb{Z}_p^3$. Since $\tau(t\mathscr{O}_p) = \sum_{ij} t_{ij}\mathbb{Z}_p = p^\alpha\mathbb{Z}_p$ by assumption, we have $\mathrm{mes}(D'_{00}) = \mathrm{mes}(A^{(p)}_{S_0,\Delta}(\tau(t\mathscr{O}_p))$. Assume $|t_{12}|_p > p^{-\alpha}$. Then the condition (5.9) reduces to the two conditions ${}^tX_2T_0X_1 - \Delta \in t_{12}\mathbb{Z}_p$ and $x_{21} - t_{12}^{-1}({}^tX_2T_0X_1 - \Delta) \in p^\alpha t_{12}^{-1}\mathbb{Z}_p$ for $(X_1, X_2, x_{11}, x_{12}, x_{21}) \in \mathbb{Z}_p^{2q} \oplus \mathbb{Z}_p^{2q} \times \mathbb{Z}_p^3$. Since $\tau(t\mathscr{O}_p) = t_{12}\mathbb{Z}_p$, we have $\mathrm{mes}(D'_{00}) = |t_{12}p^{-\alpha}|_p^{-1}\mathrm{mes}(A^{(p)}_{S_0,\Delta}(\tau(t\mathscr{O}_p)))$. This completes the proof for the case $\beta \geqslant \alpha$ in (b). We omit the consideration for the case $\beta < \alpha$, since it is similar as above. □

5.3.2. *The functions $\varphi^{(p)}$.* We consider the functions $\varphi^{(p)}(g_0\,;\,g)$ with $g_0 \in \mathfrak{X}_{\mathbb{Q}}$.

PROPOSITION 14. *For $X \in \mathscr{O}_p^q$, let $L^{(p)}(X) = \{\lambda \in E_p \mid \mathscr{O}_p\,\lambda + S_0(X, \mathscr{O}_p^q) = \mathscr{O}_p\}$. Then for $g = \mathsf{m}(t\,;\,m_0)$ with $(t, m_0) \in E_p^\times \times \mathsf{G}_0(\mathbb{Q}_p)$, we have*

$$\varphi^{(p)}(g_0\,;\,g) = |\Delta|_p^{-\delta}|\mathrm{N}(t)|_p^{\delta/2}\mathrm{mes}(L^{(p)}(m_0^{-1}g_0^{-1}x_0))\,.$$

If $(g_0m_0)^{-1}x_0 \notin S_0^{-1}\mathscr{O}_p^q$, then $\varphi^{(p)}(g_0\,;\,g) = 0$.

6. Mellin transform of constant term

In this section throughout, we fix a Grössencharacter $\psi \in \hat{C}_E$ and a $\mathbb{C}$-valued function ξ on C_{S_0}.

6.1. Computation of certain Dirichlet series

Let $\{t_i\}_{i\in C_E}$ be a complete set of representatives of C_E in $E_{\mathbb{A}}^{(1)}$. The intersection $E \cap t_i(E_{\mathbb{R}} \times \prod_p \mathscr{O}_p)t_i^{-1}$ in $E_{\mathbb{A}}$ regarded as a subset of E is an order of E, which we denote by $\mathscr{O}(t_i)$. Put $\mu(\mathscr{O}) := \sum_{i\in C_E}(\#(\mathscr{O}(t_i)^\times))^{-1}$. For each $i \in C_E$, we write $t_i = (t_{i,\infty}, t_{i,\mathrm{f}})$ with $t_{i,\mathrm{f}} \in E_{\mathbb{A}_\mathrm{f}}^\times$ and $t_{i,\infty} \in E_{\mathbb{R}}^\times$. Put $\mathscr{O}_{\mathbb{A}} = E_{\mathbb{R}} \times \prod_p \mathscr{O}_p$.

In this subsection, we shall consider the Dirichlet series

$$L_{S_0,\Delta}(t_i\,;\,\nu) := \sum_{n\in E^\times\cap(\mathscr{O}_{\mathbb{A}}t_i^{-1})} \frac{\Phi^{(\mathrm{f})}(n\,;\,\mathsf{m}(t_{i,\mathrm{f}}\,;\,\mathtt{I}_q))}{|\mathrm{N}(n_\infty t_{i,\infty})|_\infty^{(\nu+2\rho_0-\delta)/2}}\,, \quad \nu \in \mathbb{C} \tag{6.1}$$

with $\Phi^{(\mathrm{f})}(n\,;\,g) := \prod_p \Phi^{(p)}(n\,;\,g_p)$, $g = (g_p) \in \mathsf{G}(\mathbb{A}_\mathrm{f})$, or rather their average

$$L_{S_0,\Delta}(\nu\,;\,\psi) := \mu(\mathscr{O})^{-1}\sum_{i\in C_E}\frac{|\mathrm{N}(t_{i,\infty})|_\infty^{\rho_0}\psi(t_i)}{\#(\mathscr{O}(t_i)^\times)}L_{S_0,\Delta}(t_i\,;\,\nu) \tag{6.2}$$

mainly. From now on we assume $2 \notin R(E)$ if $\delta = 2$. We also make an assumption on the integer Δ and keep it from now on, until the end of this article:

$$|\Delta|_p = 1\,, \quad \forall p \in S(E)\,. \tag{6.3}$$

The following lemma is standard.

LEMMA 13. *For each $p \in I(\mathbb{Q})$, let $\mathfrak{R}_p$ be the set of the right ideals $\mathscr{A}_p(\neq 0)$ of $\mathscr{O}_p$. Let $\mathfrak{R}_{\mathbb{A}}$ be the set of all those families $\mathscr{A} = \{\mathscr{A}_p\}_{p\in I(\mathbb{Q})}$ such that $\mathscr{A}_p \in \mathfrak{R}_p$ at each p and $\mathscr{A}_p = \mathscr{O}_p$ at almost all p. Let $i \in C_E$. For each $\mathscr{O}(t_i)^\times$-coset $[n] = n\mathscr{O}(t_i)^\times$ contained in $E^\times \cap (\mathscr{O}_{\mathbb{A}}t_i^{-1})$, put $\mathscr{A}_i([n]) := \{nt_{i,p}\mathscr{O}_p\}_{p\in I(\mathbb{Q})}$, an element of $\mathfrak{R}_{\mathbb{A}}$. Then the assignment $(i, n) \mapsto \mathscr{A}_i([n])$ sets up a bijection from the set of all pairs (i, n) with $i \in C_E$, $[n] \in (E^\times \cap (\mathscr{O}_{\mathbb{A}}t_i^{-1}))/\mathscr{O}(t_i)^\times$ onto the set $\mathfrak{R}_{\mathbb{A}}$.*

DEFINITION 2. Let p be a prime. For $D \in \mathbb{Q}_p$, set

$$\Psi_{S_0,D}^{(p)}(\nu) := \sum_{e=0}^{\infty}\mathrm{mes}(A_{S_0,D}^{(p)}(p^e\mathbb{Z}_p))\,p^{-e(\nu-1)}\,, \quad \nu \in \mathbb{C}\,.$$

We remark that the trivial estimate $\mathrm{mes}(A^{(p)}_{S_0,D}(p^e\mathbb{Z}_p)) \leqslant \mathrm{mes}(S_0^{-1}\mathscr{O}_p^q)$ $(\forall e \in \mathbb{N})$ gives a region of absolute convergence $\mathrm{Re}(\nu) > 1$ for the series $\Psi^{(p)}_{S_0,D}(\nu)$.

PROPOSITION 15. *For each $i \in C_E$, the Dirichelt series $L_{S_0,\Delta}(t_i\,;\nu)$ converges absolutely and locally uniformly on* $\mathrm{Re}(\nu) > \delta$. *For $p \in I(\mathbb{Q}) - S(E)$, set*

$$\Xi^{(p)}_{S_0,\Delta}(\nu) := \Psi^{(p)}_{S_0,\Delta}(\nu) - |\mathrm{N}(\varpi)|_p^{\nu/2+\rho_0-\delta/2}\Psi^{(p)}_{S_0,\Delta\mathrm{N}(\varpi)^{-1}}(\nu)\,, \quad \nu \in \mathbb{C} \tag{6.4}$$

with ϖ any prime element of $\mathscr{O}_p$. For $p \in S(E)$, set

$$\Xi^{(p)}_{S_0,\Delta}(\nu) := \Psi^{(p)}_{S_0,\Delta}(\nu)\,, \quad \nu \in \mathbb{C}\,.$$

We have an Euler product expression of $L_{S_0,\Delta}(\nu\,;\psi)$ absolutely convergent on $\mathrm{Re}(\nu) > \delta$*:*

$$\begin{aligned} L_{S_0,\Delta}(\nu\,;\psi) &= \mu(\mathscr{O})^{-1}D_{\mathscr{O}}^{\nu/\delta}\psi(\mathfrak{d}_E)^{-1} \\ &\quad\times \prod_{p\in S(E)\cup R(E)}(1-p^{-\nu})L^{(p)}(\nu/2\,;\lambda^{(p)}_\psi)\prod_{p\in I(\mathbb{Q})}\Xi^{(p)}_{S_0,\Delta}(\nu)\,. \end{aligned} \tag{6.5}$$

Proof. For $p \in I(\mathbb{Q}) - S(E)$, $\mathscr{A}_p = t\mathscr{O}_p \in \mathfrak{R}_p$, we put $\mathfrak{M}^{(p)}(\mathscr{A}_p) := |2|_p^{1-\delta}|\mathrm{N}(t)|_p^{-\delta/2}$ $\mathrm{mes}(D^{(p)}(t))$. By Proposition 12 and Proposition 13, this is independent of t. For $\mathscr{A} = \{\mathscr{A}_p\} \in \mathfrak{R}_{\mathbb{A}}$, we set $\mathfrak{M}(\mathscr{A}) := \prod_{p\in I(\mathbb{Q})}\mathfrak{M}^{(p)}(\mathscr{A}_p)$. (Note that if $p \notin R(E)$ is relatively prime to $2\Delta\det(Q_0)$ then $\mathfrak{M}^{(p)}(\mathscr{O}_p) = 1$.) Let $\mathscr{A}_i([n]) = \{\mathscr{A}^{(p)}\}$ with $i \in C_E$ and $[n] \in (E^\times \cap \mathscr{O}_{\mathbb{A}}t_i^{-1})/\mathscr{O}(t_i)^\times$. Then by Proposition 11 and also by $|\mathrm{N}(n)|_{\mathbb{A}} = 1$, $t_i \in E^{(1)}_{\mathbb{A}}$, we have

$$\begin{aligned} \Phi^{(\mathrm{f})}(n\,;\mathsf{m}(t_{i,\mathrm{f}}\,;\mathtt{I}_q)) &= \prod_{p\in I(\mathbb{Q})}(|\mathrm{N}(t_{i,p})|_p^{\rho_0}|\mathrm{N}(nt_{i,p})|_p^{-\rho_0+\delta/2}\mathfrak{M}^{(p)}(\mathscr{A}_p)) \\ &= |N(t_{i,\infty})|_\infty^{-\rho_0}|\mathrm{N}(nt_{i,\infty})|_\infty^{\rho_0-\delta/2}\mathfrak{M}(\mathscr{A}_i([n]))\,. \end{aligned}$$

Hence

$$L_{S_0,\Delta}(t_i\,;\nu) = \sum_{[n]\in(E^\times\cap\mathscr{O}_{\mathbb{A}}t_i^{-1})/\mathscr{O}(t_i)^\times}\#(\mathscr{O}(t_i)^\times)\cdot|\mathrm{N}(t_{i,\infty})|_\infty^{-\rho_0}\frac{\mathfrak{M}(\mathscr{A}_i([n]))}{|\mathrm{N}(nt_{i,\infty})|_\infty^{\nu/2}}\,.$$

In the computations below we use identification $\mathfrak{R}_{\mathbb{A}} \cong E_{\mathrm{f}}^\times/\mathscr{O}_{\mathrm{f}}^\times$ to consider the function $\mathfrak{M}$ on $E_{\mathrm{f}}^\times$ and the functions ψ, $\tilde{\psi}$ (defined in Lemma 2) on $\mathfrak{R}_{\mathbb{A}}$. Setting aside the problem of convergence for a while, by using Lemma 13, we have

$$\begin{aligned} \mu(\mathscr{O})L_{S_0,\Delta}(\nu\,;\psi) &= \sum_{i\in C_E}\Bigg(\sum_{[n]\in(E^\times\cap\mathscr{O}_{\mathbb{A}}t_i^{-1})/\mathscr{O}(t_i)^\times}\frac{\mathfrak{M}(\mathscr{A}_i([n]))\psi(t_i)}{|\mathrm{N}(nt_{i,\infty})|_\infty^{\nu/2}}\Bigg) \\ &= \sum_{\mathscr{A}\in\mathfrak{R}_{\mathbb{A}}}\psi(\mathscr{A})\mathfrak{M}(\mathscr{A})|\mathrm{N}(\mathscr{A})|_p^{\nu/2} \\ &= \int_{E_{\mathrm{f}}^\times}\psi(x_{\mathrm{f}})\mathfrak{M}(x_{\mathrm{f}})|\mathrm{N}(x_{\mathrm{f}})|_{\mathrm{f}}^{\nu/2}\,d^\times x_{\mathrm{f}} \\ &= \int_{E_{\mathrm{f}}^\times}\tilde{\psi}(x_{\mathrm{f}})\mathfrak{M}(x_{\mathrm{f}})|\mathrm{N}(x_{\mathrm{f}})|_{\mathrm{f}}^{\nu/2}\,d^\times x_{\mathrm{f}} \end{aligned} \tag{6.6}$$

$$= \sum_{\mathscr{A} \in \mathfrak{R}_{\mathbb{A}}} \tilde{\psi}(\mathscr{A}) \mathfrak{M}(\mathscr{A}) |\mathrm{N}(\mathscr{A})|_p^{\nu/2}.$$

Here to obtain the fourth equality above, we use the relation $\mathfrak{M}(u_{\mathrm{f}} x) = \mathfrak{M}(x)$ $(\forall u_{\mathrm{f}} \in \mathscr{O}_{\mathrm{f}}^{\times})$, which follows from the fact that $\mathfrak{M}(\mathscr{A})$ is determined only by the traces $\tau(\mathscr{A}_p)$ combined with the obvious relation $\tau(u_p \mathscr{A}_p) = \tau(\mathscr{A}_p u_p) = \tau(\mathscr{A}_p)$ $(\forall u_p \in \mathscr{O}_p^{\times})$. By Lemma 2, we obtain $\mu(\mathscr{O}) L_{S_0,\Delta}(\nu\,;\psi) = \prod_p L^{(p)}(\nu\,;\psi)$ with

$$L^{(p)}(\nu\,;\psi) := \sum_{\mathscr{A}_p = t\mathscr{O}_p \in \mathfrak{R}_p} \frac{\mathfrak{M}^{(p)}(\mathscr{A}_p)}{\mathrm{mes}(\mathscr{O}_p^{\times} t \mathscr{O}_p^{\times})} \lambda_{\psi}^{(p)}(\mathrm{Char}_{\mathscr{O}_p^{\times} t \mathscr{O}_p^{\times}}) |\mathrm{N}(\mathscr{A}_p)|_p^{\nu/2},$$

$$b^{(p)}(f\,;\nu,\psi) := \sum_{\substack{\mathscr{A}_p = t\mathscr{O}_p \in \mathfrak{R}_p; \\ \tau(\mathscr{A}_p) = p^f \mathbb{Z}_p}} \frac{\lambda_{\psi}^{(p)}(\mathrm{Char}_{\mathscr{O}_p^{\times} t \mathscr{O}_p^{\times}})}{\mathrm{mes}(\mathscr{O}_p^{\times} t \mathscr{O}_p^{\times})} |\mathrm{N}(\mathscr{A}_p)|_p^{\nu/2}, \ (f \in \mathbb{N}).$$

Let us evaluate the series $L^{(p)}(\nu\,;\psi)$ and $b^{(p)}(f\,;\nu,\psi)$.

- First we treat the case when $p \in I(\mathbb{Q}) - S(E)$. Then since $\mathfrak{R}_p = \{\varpi^e \mathscr{O}_p \,|\, e \in \mathbb{N}\}$, by Proposition 12, it turns out $L^{(p)}(\nu\,;\psi)$ equals

$$\sum_{f=0}^{\infty} b^{(p)}(f\,;\nu,\psi) p^f \left(\mathrm{mes}(A_{S_0,\Delta}^{(p)}(p^f \mathbb{Z}_p)) - |\mathrm{N}(\varpi)|_p^{\nu/2+\rho_0-\delta/2} \mathrm{mes}(A_{S_0,\Delta \mathrm{N}(\varpi)^{-1}}^{(p)}(p^f \mathbb{Z}_p))\right).$$

  We have two cases.

  - Suppose $p \in I(E)$. Then we may assume $\varpi = p$ and $\tau(\varpi^e \mathscr{O}_p) = p^e \mathbb{Z}_p$ $(\forall e \in \mathbb{N})$, unless $E = \mathbb{Q}$ and $p = 2$. This gives $b^{(p)}(f\,;\nu,\psi) = p^{-f\nu}$, and therefore $L^{(p)}(\nu\,;\psi) = \Xi_{S_0,\Delta}^{(p)}(\nu)$. When $E = \mathbb{Q}$ and $p = 2$, we still have $\varpi = 2$ but $\tau(\varpi^e \mathscr{O}_p) = 2^{e+1} \mathbb{Z}_p$ $(\forall e \in \mathbb{N})$. This gives $b^{(p)}(f\,;\nu,\psi) = 2^{-(f-1)\nu}$, and hence $L^{(2)}(\nu\,;\psi) = 2^{\nu} \Xi_{S_0,\Delta}^{(2)}(\nu)$ in this case.
  - Suppose $p \in R(E)$. Let $\varpi^{d_p} \mathscr{O}_p$ be the 'different' of $\mathscr{O}_p$. (Note $d_p = 1$ if E_p is a quaternion algebra.) Then we have $\tau(\varpi^e \mathscr{O}_p) = p^{[(e+d_p)/2]} \mathbb{Z}_p$ for each $e \in \mathbb{N}$. Hence the condition $\tau(\varpi^e \mathscr{O}_p) = p^f \mathbb{Z}_p$ is equivalent to $e = 2f - d_p,\ 2f - d_p + 1$. Put $\lambda_p = \lambda_{\psi}^{(p)}(\mathrm{Char}_{\mathscr{O}_p^{\times} \varpi \mathscr{O}_p^{\times}})$ to simplify notation. Then noting Lemma 3, we have $b^{(p)}(f\,;\nu,\psi) = p^{-(2f-d_p)\nu/2} \lambda_p^{2f-d_p} + p^{-(2f-d_p+1)\nu/2} \lambda_p^{2f-d_p+1} = \lambda_p^{-d_p} p^{d_p \nu/2} p^{-f\nu}$ $(1 + \lambda_p p^{-\nu/2})$, and consequently obtain $L^{(p)}(\nu\,;\psi) = \lambda_p^{-d_p}\ p^{d_p \nu/2}(1 + \lambda_p p^{-\nu/2})$ $\Xi_{S_0,\Delta}^{(p)}(\nu)$.

- We treat the case when $p \in S(E)$. By Proposition 13,

$$L^{(p)}(\nu\,;\psi) = \sum_{f=0}^{\infty} b^{(p)}(f\,;\nu,\psi) p^f \mathrm{mes}(A_{S_0,\Delta}^{(p)}(p^f \mathbb{Z}_p)).$$

  We have two cases.

– Assume $E_p = \mathbb{Q}_p \oplus \mathbb{Q}_p$, $\mathcal{O}_p = \mathbb{Z}_p \oplus \mathbb{Z}_p$. For $(l_1, l_2) \in \mathbb{N}^2$, put $\mathcal{A}(l_1, l_2) := p^{l_1}\mathbb{Z}_p \oplus p^{l_2}\mathbb{Z}_p$. Then the map $(l_1, l_2) \mapsto \mathcal{A}(l_1, l_2)$ sets up a bijection from $\mathbb{N}^2$ onto $\mathfrak{R}_p$, and $\tau(\mathcal{A}(l_1, l_2)) = p^{\inf(l_1,l_2)}\mathbb{Z}_p$. Hence the ideals $\mathcal{A}_p$ such that $\tau(\mathcal{A}_p) = p^f\mathbb{Z}_p$ are listed as $p^f\mathcal{A}(l_1, 0)$ $(l_1 \in \mathbb{N}^*)$, $p^f\mathcal{A}(0, l_2)$ $(l_2 \in \mathbb{N}^*)$ and $\mathcal{A}(f, f)$. Put $\lambda_{1p} = \lambda_\psi^{(p)}(\mathrm{Char}_{\mathcal{O}_p^\times(p,1)\mathcal{O}_p^\times})$, $\lambda_{2p} = \lambda_\psi^{(p)}(\mathrm{Char}_{\mathcal{O}_p^\times(1,p)\mathcal{O}_p^\times})$ to simplify notation. Then noting Lemma 3 we have

$$\begin{aligned} b^{(p)}(f\,;\nu,\psi) &= \sum_{l_1=1}^{\infty} p^{-(2f+l_1)\nu/2}\lambda_{1p}^{l_1} + \sum_{l_2=1}^{\infty} p^{-(2f+l_2)\nu/2}\lambda_{2p}^{l_2} + p^{-f\nu} \\ &= p^{-f\nu}\frac{1-p^{-\nu}}{(1-\lambda_{1p}\,p^{-\nu/2})(1-\lambda_{2p}\,p^{-\nu/2})}\,. \end{aligned}$$

Therefore $L^{(p)}(\nu\,;\psi) = (1-p^{-\nu})L^{(p)}(\nu/2\,;\lambda_\psi^{(p)})\Xi_{S_0,\Delta}^{(p)}(\nu)$.

– Assume $E_p = \mathrm{M}_2(\mathbb{Q}_p)$, $\mathcal{O}_p = \mathrm{M}_2(\mathbb{Z}_p)$. For $(l_1, l_2) \in \mathbb{N}^2$, $x \in R_{l_2}$, put $\mathcal{A}(l_1, l_2\,; x) := \begin{bmatrix} p^{l_1} & 0 \\ x & p^{l_2} \end{bmatrix}\mathrm{M}_2(\mathbb{Z}_p)$. Here R_l denotes a complete set of representatives of $\mathbb{Z}_p/p^l\mathbb{Z}_p$. Then the map $(l_1, l_2\,; x) \mapsto \mathcal{A}(l_1, l_2\,; x)$ is a bijection onto $\mathfrak{R}_p$, and $\tau(\mathcal{A}(l_1, l_2\,; x)) = p^{l_1}\mathbb{Z}_p + p^{l_2}\mathbb{Z}_p + x\mathbb{Z}_p$, $|\mathrm{N}(\mathcal{A}(l_1, l_2\,; x))| = p^{l_1+l_2}$. Hence the right ideal $\mathcal{A}_p \in \mathfrak{R}_p$ such that $\tau(\mathcal{A}_p) = p^f\mathbb{Z}_p$ is one of the following form.

$$\begin{aligned} &p^f\mathcal{A}(0, l_2\,; x)\ (l_2 \in \mathbb{N}^*,\ x \in R_{l_2})\,, \quad p^f\mathcal{A}(l_1, 0\,; 0)\ (l_1 \in \mathbb{N}^*)\,, \\ &p^f\mathcal{A}(l_1, l_2\,; y)\ (l_1,\ l_2 \in \mathbb{N}^*,\ y \in R_{l_2} \cap \mathbb{Z}_p^\times)\,, \quad p^f\mathcal{O}_p\,. \end{aligned}$$

Put $\phi_{k,l} = \mathrm{Char}_{\mathcal{O}_p^\times\left[\begin{smallmatrix} p^k & 0 \\ 0 & p^l \end{smallmatrix}\right]\mathcal{O}_p^\times}$ to simplify notation. Then by Lemma 3, $\lambda_\psi^{(p)}(\phi_{k,l}) = \lambda_\psi^{(p)}(\phi_{k-l,0})$ for every $(k, l) \in \mathbb{Z}^2$ with $k \geqslant l$. Since $\mathcal{O}_p^\times\begin{bmatrix} p^k & 0 \\ 0 & p^l \end{bmatrix}\mathcal{O}_p^\times$ $(k > l)$ is a disjoint union of $p^{k-l+1}(p+1)$ left $\mathcal{O}_p^\times$-cosets, its measure is $p^{k-l+1}(p+1)$. From these,

$$\begin{aligned} b^{(p)}(f\,;\nu,\psi) &= \sum_{l_2=1}^{\infty}\frac{\lambda_\psi^{(p)}(\phi_{l_2,0})}{p^{l_2-1}(p+1)}p^{l_2}p^{-(2f+l_2)\nu/2} + \sum_{l_1=1}^{\infty}\frac{\lambda_\psi^{(p)}(\phi_{l_1,0})}{p^{l_1-1}(p+1)}p^{-(2f+l_1)\nu/2} \\ &\quad + \sum_{l_1,l_2\geqslant 1}\frac{\lambda_\psi^{(p)}(\phi_{l_1+l_2,0})}{p^{l_1+l_2-1}(p+1)}\frac{p^{l_2}-p^{l_2-1}}{p^{(l_1+l_2+2f)\nu/2}} + p^{-f\nu} \\ &= p^{-f\nu}h(\nu/2\,;\lambda_\psi^{(p)}) \end{aligned}$$

after an elementary computation. Hence, by Lemma 4, we have $L^{(p)}(\nu\,;\psi) = (1-p^{-\nu})L^{(p)}(\nu/2\,;\lambda_\psi^{(p)})\Xi_{S_0,\Delta}^{(p)}(\nu)$.

This completes the evaluation of the series $L^{(p)}(\nu\,;\psi)$. It remains to discuss the convergence of the Euler product (6.6). Since ψ is bounded, the series $L_{S_0,\Delta}(\nu\,;\psi)$ $(\mathrm{Re}(\nu) \geqslant \sigma)$ is bounded by a constant multiple of $L_{S_0,\Delta}(\sigma\,;1)$. The product $\prod_{p\in S(E)}(1+p^{-\sigma/2})(1-p^{-\sigma/2+\delta/2-1})^{-1}$ is trivial if $E = \mathbb{Q}$ and is convergent for $\sigma > 2$, a fortiori for $\sigma > \delta$ if

$\delta \geqslant 2$. This shows the convergence of $\prod_p(1-p^{-\sigma})L^{(p)}(\sigma/2\,;\lambda_1^{(p)})$. Hence the convergence of (6.6) follows from $\prod_p \Xi^{(p)}_{S_0,\Delta}(\sigma) < \infty\ (\sigma > 1)$, which in turn is a consequence of the explicit formula of good local factors $\Xi^{(p)}_{S_0,\Delta}(\sigma)$ to be proved in Proposition 16. To obtain the formula (6.5), we should remark $D_{\mathscr{O}} = \prod_{p\in R(E)} p^{\delta d_p/2}$ if $\delta > 1$ and $D_{\mathbb{Z}} = 2$. We also note $\prod_{p\in R(E)} \lambda_\psi^{(p)}(\mathrm{Char}_{\mathscr{O}_p^\times \varpi \mathscr{O}_p^\times}) = \psi(\mathfrak{d}_E)$, and $(1-p^{-\nu})L^{(p)}(\nu/2\,;\lambda_\psi^{(p)}) = 1$ if $p \in I(E)$, $(1-p^{-\nu})L^{(p)}(\nu/2\,;\lambda_\psi^{(p)}) = (1+\lambda_\psi^{(p)}(\mathrm{Char}_{\mathscr{O}_p^\times \varpi \mathscr{O}_p^\times})p^{-\nu/2})$ if $p \in R(E)$. □

6.2. Evaluation of good local factors

We evaluate $\Xi^{(p)}_{S_0,\Delta}(\nu)$ at the primes p relatively prime to $2\Delta\det(Q_0)$. We first have a lemma.

LEMMA 14. *Let p be an odd prime and l a positive integer. Let $Q = {}^tQ \in \mathrm{M}_l(\mathbb{Z}_p)$ be a symmetric matrix such that $|\det Q|_p = 1$. For a given $D \in \mathbb{Z}_p^\times$ and $e \in \mathbb{N}^*$, put $A_D(e) := \{X \in \mathbb{Z}_p^l \mid Q[X] - D \in p^e\mathbb{Z}_p\}$.*

(1) *If we put the Haar measure on $\mathbb{Z}_p^l$ such that* $\mathrm{mes}(\mathbb{Z}_p^l) = 1$, *we have* $\mathrm{mes}(A_D(e)) = p^{-le}r_e$ *with r_e the cardinality of the set*

$$\tilde{A}_D(e) := \{\xi \in \mathbb{Z}_p^l/p^e\mathbb{Z}_p^l \mid Q[\xi] \equiv D \pmod{p^e}\}.$$

(2) *Let $e > 1$. Then the natural reduction-map $\mathbb{Z}_p^l/p^e\mathbb{Z}_p^l \to \mathbb{Z}_p^l/p^{e-1}\mathbb{Z}_p^l$ induces a map $f_e : \tilde{A}_D(e) \to \tilde{A}_D(e-1)$. Any fibre of the map f_e has p^{l-1} elements.*

(3) *For $e \in \mathbb{N}^*$,*

$$r_e = p^{(l-1)(e-1)}\begin{cases} p^{l-1} - (-1)^{\frac{l}{2}\frac{p-1}{2}}\left(\frac{\det Q}{p}\right)p^{l/2-1}, & (l \equiv 0 \pmod 2), \\ p^{l-1} + (-1)^{\frac{l-1}{2}\frac{p-1}{2}}\left(\frac{D\,\det Q}{p}\right)p^{(l-1)/2}, & (l \equiv 1 \pmod 2). \end{cases}$$

(4) *For $e \in \mathbb{N}^*$,*

$$\mathrm{mes}(A_D(e)) = p^{-e}\begin{cases} 1 - (-1)^{\frac{l}{2}\frac{p-1}{2}}\left(\frac{\det Q}{p}\right)p^{-l/2}, & (l \equiv 0 \pmod 2), \\ 1 + (-1)^{\frac{l-1}{2}\frac{p-1}{2}}\left(\frac{D\,\det Q}{p}\right)p^{(1-l)/2}, & (l \equiv 1 \pmod 2). \end{cases}$$

Proof. (1), (2) are easily proved. We have (3) from (2) once r_1 is known; the formula of r_1 is given in [9, §7,Theorem 5 (p. 103)]. (4) follows from (1) and (3). □

PROPOSITION 16. *Assume δq is even. Let χ_{D_0} be the Kronecker character of the quadratic field $\mathbb{Q}(\sqrt{D_0})$ with $D_0 := (-1)^{\frac{\delta q}{2}}\det Q_0$. Then we have*

$$\Xi^{(p)}_{S_0,\Delta}(\nu) = (1 - \chi_{D_0}(p)p^{-\nu-\rho_0+\delta-1})(1-p^{-\nu})^{-1}, \quad \mathrm{Re}(\nu) > 1.$$

Assume δq is odd, that is $\delta = 1$ and q is odd. Let χ_{D_0} be the Kronecker character of the quadratic field $\mathbb{Q}(\sqrt{D_0})$ with $D_0 := (-1)^{\frac{q-1}{2}}\Delta\det Q_0$. Then we have

$$\Xi^{(p)}_{S_0,\Delta}(\nu) = (1 + \chi_{D_0}(p)p^{-\nu+\frac{1-q}{2}})(1-p^{-\nu})^{-1}, \quad \mathrm{Re}(\nu) > 1.$$

Proof. The assumption $p \nmid 2\Delta\det(Q_0)$ means $p \neq 2$, $S_0^{-1}\mathscr{O}_p^q = \mathscr{O}_p^q$ and $\mathrm{N}(\varpi)^{-1}\Delta \notin \mathbb{Z}_p$. Since $S_0[X] = 2^{-1}Q_0[X]$ is integral on $\mathscr{O}_p^q$, we have $A^{(p)}_{S_0,\Delta\mathrm{N}(\varpi)^{-1}}(p^e\mathbb{Z}_p) = \varnothing$ ($\forall e \in$

$\mathbb{N}$). Hence the second term of $\Xi^{(p)}_{S_0,\Delta}(\nu)$ given by (6.4) is zero. Thus to obtain $\Xi^{(p)}_{S_0,\Delta}(\nu)$, we have only to compute $\Psi^{(p)}_{S_0,\Delta}(\nu)$. When δq is even, we apply Lemma 14 for the quadratic form $Q = Q_0$ on $\mathscr{O}_p^q \cong \mathbb{Z}_p^{\delta q}$ to have $\mathrm{mes}(A^{(p)}_{S_0,\Delta}(p^e\mathbb{Z}_p)) = p^{-e}(1-\chi_{D_0}(p)p^{-\delta q/2})$, $(\forall e \in \mathbb{N}^*)$. Obviously $\mathrm{mes}(A^{(p)}_{S_0,\Delta}(\mathbb{Z}_p)) = \mathrm{mes}(\mathscr{O}_p^q) = 1$. Therefore

$$\Psi^{(p)}_{S_0,\Delta}(\nu) = \sum_{e=1}^{\infty}(1-\chi_{D_0}(p)p^{-\delta q/2})p^{-e\nu} + 1 = (1-\chi_{D_0}(p)p^{-\nu-\delta q/2})(1-p^{-\nu})^{-1}\,.$$

The case when δq is odd is similar. □

6.3. Bad local factors

We consider $\Xi^{(p)}_{S_0,\Delta}(\nu)$ for $p|2\Delta\det(Q_0)$.

LEMMA 15. *Let p be a prime dividing $2\Delta\det(Q_0)$. Then there exist polynomials $P^{(p)}_{S_0,\Delta}(X)$ and $P^{(p)}_{S_0,\Delta\mathrm{N}(\varpi)^{-1}}(X)$ such that*

$$\Xi^{(p)}_{S_0,\Delta}(\nu) =(P^{(p)}_{S_0,\Delta}(p^{-\nu}) - |\mathrm{N}(\varpi)|_p^{\nu/2-\delta/2+\rho_0}P^{(p)}_{S_0,\mathrm{N}(\varpi)^{-1}\Delta}(p^{-\nu}))((1-p^{-\nu})^{-1}\,,\qquad \mathrm{Re}(\nu) > 1\,.$$

Proof. First suppose $p \neq 2$. From the assumption (6.3), E_p is necessarily a division algebra for $p|\Delta\det(Q_0)$. For $e \in \mathbb{N}$, let r_e be the cardinality of the set $\{X \in S_0^{-1}\mathscr{O}_p^q/p^e\mathscr{O}_p^q|\, S_0[X] \equiv \Delta \pmod{p^e}\}$. Then $\mathrm{mes}(A^{(p)}_{S_0,\Delta}(p^e\mathbb{Z}_p)) = p^{-e\delta q}r_e$. By [11, Lemma 5.6.4 (p. 96)], there exists $e_0 = e_0(S_0,\Delta) \in \mathbb{N}$ such that $c^{(p)}_{S_0,\Delta} := p^{-(e-1)(\delta q-1)}r_e$ is constant in $e \geqslant e_0$. Hence

$$\begin{aligned}\Psi^{(p)}_{S_0,\Delta}(\nu) &= \sum_{e=0}^{e_0-1}\mathrm{mes}(A^{(p)}_{S_0,\Delta}(p^e\mathbb{Z}_p))p^{-e(\nu-1)} + c^{(p)}_{S_0,\Delta}\,p^{1-\delta q}\sum_{e=e_0}^{\infty}p^{-e\nu}\\ &= P^{(p)}_{S_0,\Delta}(p^{-\nu})(1-p^{-\nu})^{-1}\end{aligned}$$

with $P^{(p)}_{S_0,\Delta}(X) := (1-X)\sum_{e=0}^{e_0-1}\mathrm{mes}(A^{(p)}_{S_0,\Delta}(p^e\mathbb{Z}_p))p^eX^e + c^{(p)}_{S_0,\Delta}p^{1-\delta q}X^{e_0}$. From this we have the desired formula. The case of $p = 2$ is similar. □

6.4. Estimate of constant term

Let us estimate the constant term $F_{s,(\mathsf{P})}(\mathsf{m}(t\,;g_0))$ for large $|\mathrm{N}(t)|_{\mathbb{A}}$.

PROPOSITION 17. *Given a compact set $B \subset \{s \in \mathbb{C}|\,\mathrm{Re}(s) > \rho_0\}$ and a large positive number $R > 0$, there exists a constant $C > 0$ and $\sigma > \rho_0$ such that*

$$\sum_{n\in E^\times}|\Phi_s(n\,;\mathsf{m}(t\,;m_0))| \leqslant C\,|\mathrm{N}(t)|_{\mathbb{A}}^{(-\sigma+\rho_0)/2}\,, \tag{6.7}$$

$$\sum_{g_0\in\mathfrak{X}_{\mathbb{Q}}}|\varphi_s(g_0\,;\mathsf{m}(t\,;m_0))| \leqslant C\,(\#\{X\in\mathscr{L}_0(m_0)^*|\,S_0[X]=\Delta\})|\mathrm{N}(t)|_{\mathbb{A}}^{\delta/2}\,, \tag{6.8}$$

$$\forall s\in B,\ \forall(t,m_0)\in E_{\mathbb{A}}^\times\times\mathsf{G}_0(\mathbb{A}),\ |\mathrm{N}(t)|_{\mathbb{A}} > R\,.$$

Here for $m_0 \in \mathsf{G}_0(\mathbb{A})$, $\mathscr{L}_0(m_0)$ *is the* $\mathscr{O}$*-lattice in* E^q *determined by the condition* $\mathscr{L}_0(m_0)\mathscr{O}_p = m_{0,p}\mathscr{O}_p^q$ $(\forall p \in I(\mathbb{Q}))$ *and* $\mathscr{L}_0(m_0)^*$ *means its dual lattice with respect to* S_0.

Proof. Since G_0 is $\mathbb{R}$-anisotropic, $|\det(m_0|E^q_{\mathbb{A}})|_{\mathbb{A}} = |\det(m_{0,\infty}|E^q_{\mathbb{R}})|_\infty = 1$ $(\forall m_0 \in \mathsf{G}_0(\mathbb{A}))$. Hence $\Phi_s(n\,;\mathsf{m}(t\,;m_0))$ is independent of $m_0 \in \mathsf{G}_0(\mathbb{A})$ by Proposition 7 and Proposition 11. From Proposition 6, Corollary 1 and Proposition 8, the function $|\Phi_s(\mathsf{m}(t_i r\,;m_0))|$ is bounded uniformly in $s \in B$, $r > R$ by a constant multiple of

$$\mu(\mathscr{O})^{-1}\sum_{i\in C_E}\frac{\|t_{i,\infty}\|^{2\rho_0}}{\#(\mathscr{O}(t_i)^\times)}\sum_{n\in E^\times\cap\mathscr{O}_{\mathbb{A}}t_i^{-1}}\frac{\Phi^{(\mathrm{f})}(n\,;\mathsf{m}(t_{i,\mathrm{f}}\,;\mathtt{1}_q))}{\|nt_{i,\infty}\|^{\mathrm{Re}(s)+\rho_0}}\,r^{-\mathrm{Re}(s)+\rho_0}$$
$$= L_{S_0,\Delta}(\mathrm{Re}(s)-\rho_0+\delta\,;1)\cdot r^{-\mathrm{Re}(s)+\rho_0}\,.$$

Since $L_{S_0,\Delta}(\mathrm{Re}(s)-\rho_0+\delta\,;1)$ is finite (Proposition 15), this gives the estimation (6.7). We omit the proof of (6.8), since it is similar to but much easier than (6.7). □

6.5. Computation of Mellin transform of Φ_s

In this subsection we study the average of the truncated Mellin transformations of the functions $\Phi_s(\mathsf{m}(t\,;\mathtt{1}_q))$ defined by

$$\widehat{\Phi}^{[T,T']}(s,\nu,\psi) := \mu(\mathscr{O})^{-1}\sum_{i\in C_E}\frac{\psi(t_i)}{\#(\mathscr{O}(t_i)^\times)}\int_T^{T'}\Phi_s(\mathsf{m}(t_i r\,;\mathtt{1}_q))r^{\nu-\rho_0}\frac{dr}{r}$$

with truncation parameters $0 \leqslant T \leqslant T' \leqslant +\infty$.

PROPOSITION 18. (1) *For a finite* $T > 0$, *the integral* $\widehat{\Phi}^{[T,+\infty]}_{S_0,\Delta}(s,\nu,\psi)$ *converges absolutely and locally uniformly on the domain* $\mathrm{Re}(s) > \mathrm{Re}(\nu)$.

(2) *The integral* $\widehat{\Phi}^{[0,+\infty]}(s,\nu,\psi)$ *converges absolutely and locally uniformly on the domain* $\mathrm{Re}(s) > \mathrm{Re}(\nu) > \rho_0$ *and*

$$\widehat{\Phi}^{[0,+\infty]}(s,\nu,\psi) = \gamma_{S_0,\Delta}(s,\nu)\,L_{S_0,\Delta}(\nu-\rho_0+\delta\,;\psi)$$

holds there.

Proof. (1) This follows from the estimate (6.7).

(2) First we have $|\phi_s(g)| \leqslant \phi_{\mathrm{Re}(s)}(g)$ $(\forall g \in \mathsf{G}(\mathbb{A}))$ immediately from definition. This estimate in turn gives $|I_s(\gamma\,;g)| \leqslant I_{\mathrm{Re}(s)}(\gamma\,;g)$ $(\forall\gamma \in \mathsf{G}(\mathbb{Q}),\ \forall g \in \mathsf{G}(\mathbb{A}))$ (see Lemma 8). Inparticular, we obtain $|\Phi_s(n\,;g)| \leqslant \Phi_{\mathrm{Re}(s)}(n\,;g)$ $(\forall n \in E^\times,\ \forall g \in \mathsf{G}(\mathbb{A}))$. Therefore we can use this estimate to have the inequality of the following computation.

$$\sum_{i\in C_E}\frac{1}{\#(\mathscr{O}(t_i)^\times)}\int_0^\infty|\Phi_s(\mathsf{m}(t_i r\,;\mathtt{1}_q))|r^{\mathrm{Re}(\nu)-\rho_0}\frac{dr}{r}$$
$$\leqslant \sum_{i\in C_E}\frac{1}{\#(\mathscr{O}(t_i)^\times)}\int_0^\infty\Phi_{\mathrm{Re}(s)}(\mathsf{m}(t_i r\,;\mathtt{1}_q))r^{\mathrm{Re}(\nu)-\rho_0}\frac{dr}{r}$$
$$= \sum_{i\in C_E}\frac{1}{\#(\mathscr{O}(t_i)^\times)}\sum_{n\in E^\times\cap\mathscr{O}_{\mathbb{A}}t_i^{-1}}\Phi^{(\mathrm{f})}(n\,;\mathsf{m}(t_{i,\mathrm{f}}\,;\mathtt{1}_q))\widehat{\Phi}^{(\infty)}_{\mathrm{Re}(s)}(n\,;\mathrm{Re}(\nu))\|t_{i,\infty}\|^{-\mathrm{Re}(\nu)+\rho_0}$$

$$= \gamma_{S_0,\Delta}(\mathrm{Re}(s), \mathrm{Re}(\nu)) \sum_{i \in C_E} \frac{1}{\#(\mathscr{O}(t_i)^\times)} \sum_{n \in E^\times \cap \mathscr{O}_{\mathbb{A}} t_i^{-1}} \Phi^{(\mathrm{f})}(n\,;\,\mathsf{m}(t_{i,\mathrm{f}}\,;\,\mathtt{I}_q))$$
$$\times \|n\|^{-\mathrm{Re}(\nu)-\rho_0} \|t_{i,\infty}\|^{-\mathrm{Re}(\nu)+\rho_0}$$
$$= \mu(\mathscr{O}) \gamma_{S_0,\Delta}(\mathrm{Re}(s), \mathrm{Re}(\nu)) L_{S_0,\Delta}(\mathrm{Re}(\nu) - \rho_0 + \delta\,;\,1)\,.$$

The first equality is valid because the integrands above are positive. We use Proposition 9 to have the second equality when $\mathrm{Re}(s) > \mathrm{Re}(\nu) > \rho_0 - \delta$. If $\mathrm{Re}(\nu) - \rho_0 + \delta > \delta$, then $L_{S_0,\Delta}(\mathrm{Re}(\nu) - \rho_0 + \delta\,;\,1) < \infty$ by Proposition 15. This complete the proof of the convergence. By Fubini's theorem, we perform the same computation as above replacing $(\mathrm{Re}(s), \mathrm{Re}(\nu))$ by (s, ν) (and putting factors $\psi(t_i)$ in some places) to get the formula in (2). □

6.6. Computation of truncated Mellin transform of φ_s

Fix a complete set of representatives $\{m_{0,j}\}_{j \in C_{S_0}}$ of C_{S_0}. The compact space $\mathsf{G}_0(\mathbb{Q})\backslash \mathsf{G}_0(\mathbb{A})$ has a unique $\mathsf{G}_0(\mathbb{A})$-invariant measure with $\mathrm{mes}(\mathsf{G}_0(\mathbb{Q})\backslash\mathsf{G}_0(\mathbb{A})) = 1$. For each $j \in C_{S_0}$, put

$$\mu(S_0\,;\,m_{0,j}) := \mathrm{mes}(\mathsf{G}_0(\mathbb{Q})\backslash\mathsf{G}_0(\mathbb{Q}) m_{0,j} K_{0,\mathbb{A}})\,, \tag{6.9}$$

$$M_{\psi,\xi}(i,j) := \frac{\mu(S_0\,;\,m_{0,j})\,\xi(m_{0,j})\psi(t_i)}{\mu(\mathscr{O})\,\#(\mathscr{O}(t_i)^\times)}\,. \tag{6.10}$$

In this subsection we study an average of the 'truncated' Mellin transform defined by

$$\hat{\varphi}_{\psi,\xi}^{[T,T']}(s,\nu) := \sum_{i \in C_E} \sum_{j \in C_{S_0}} M_{\psi,\xi}(i,j) \int_T^{T'} \varphi_s(\mathsf{m}(t_i r\,;\,m_{0,j})) r^{\nu-\rho_0} \frac{dr}{r}$$

with truncation parameters $0 \leqslant T < T' \leqslant +\infty$.

PROPOSITION 19. *Let $0 < T_1 < T_2 < +\infty$ be positive numbers. Let $\mathrm{Re}(s) > \rho_0$. Then for any $\nu \in \mathbb{C}$, the integral $\hat{\varphi}_{S_0,\Delta}^{[T_1,T_2]}(s,\nu)$ converges absolutely and*

$$\hat{\varphi}_{\psi,\xi}^{[T_1,T_2]}(s,\nu) = C(s) M_{\psi,\xi} \frac{T_2^{\nu-\rho_0+\delta} - T_1^{\nu-\rho_0+\delta}}{\nu - \rho_0 + \delta}$$

with

$$C(s) := (2\pi\Delta^{-1})^{\delta/2} \frac{\Gamma((s+\rho_0-\delta)/2)}{\Gamma((s+\rho_0)/2))}\,,$$

and

$$M_{\psi,\xi} := \sum_{i \in C_E} \sum_{j \in C_{S_0}} M_{\psi,\xi}(i,j) \sum_{\substack{X \in \mathscr{L}_0(m_{0,j})^*;\\ S_0[X]=\Delta}} \left(\prod_{p \in I(\mathbb{Q})} \mathrm{mes}(L^{(p)}(m_{0,j}^{-1} X)) \right).$$

(*Note that, since S_0 is positive definite, the inner sum with respect to X is a finite sum.*) *The integral $\hat{\varphi}_{\psi,\xi}^{[T_1,+\infty]}(s,\nu)$ (resp. $\hat{\varphi}_{\psi,\xi}^{[0,T_2]}(s,\nu)$) is absolutely convergent if $\mathrm{Re}(\nu) < \rho_0 - \delta$ (resp. $\mathrm{Re}(\nu) > \rho_0 - \delta$).*

Proof. This follows from Proposition 14 and Proposition 10. □

7. Inner product of F_s and the Eisensten series

As before we fix a vector $\mathsf{a} \in S_0^{-1}\mathcal{O}^q$ and put $\mathsf{H} = \mathsf{G}^{[\eta]}$ and $\Delta := \Delta(\mathsf{a})$ with $\eta := \eta(\mathsf{a})$. Recall that we have been assumed that $\forall p \in S(E)$ is relatively prime to Δ. We also fix a Grössencharacter $\psi \in \hat{C}_E$ and a function $\xi : C_{S_0} \to \mathbb{C}$ and write $E(\nu : g)$ for $E(\nu\,; \psi, \xi : g)$.
In this section we show the integral

$$\langle F|E\rangle(s,\nu) := \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} F_s(g)E(\nu : g)\,d\dot{g}, \quad (s,\nu) \in \mathbb{C}^2.$$

converges in some neighborhood of the subset $\mathrm{Re}(\nu) = 0$, $\mathrm{Re}(s) > \rho_0$ in $\mathbb{C}^2$ and has a meromorphic continuation to a broader domain containing a region where both series F_s, $E(\nu)$ are absolutely convergent. Then we calculate the integral in two different ways by using differential equation and by truncation.

7.1. Calculation of inner product, via differential equation

Let Ω be a Casimir element of $\mathfrak{g}$ normalized so that its A_∞-radial part with respect to the decomposition $\mathsf{G}(\mathbb{R}) = \tilde{H}_\infty A_\infty K_\infty$ is given by

$$\frac{d^2}{dt^2} + \left((\delta-1)\frac{\cosh t}{\sinh t} + (2\rho_0 - \delta + 1)\frac{\sinh t}{\cosh t}\right)\frac{d}{dt} \tag{7.1}$$

(see [6, (Eq (4.12)]).

THEOREM 1. (1) *Let B_1 be a compact subset of $|\mathrm{Re}(\nu)| < \rho_0 - \delta$, which is disjoint from the poles of $E(\nu)$. Let B_2 be a compact subset of $\mathrm{Re}(s) > \rho_0$. Then the integral $\langle F|E\rangle(s,\nu)$ converges absolutely and uniformly on $B_2 \times B_1$.*

(2) *Let $(s,\nu) \in \mathbb{C}^2$ be such that $|\mathrm{Re}(\nu)| < \rho_0 - \delta$, $\mathrm{Re}(s) > \rho_0$. Then the formula*

$$\langle F|E\rangle(s,\nu) = \left(\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} E(\nu : h)\,d\dot{h}\right)\frac{\Gamma((s-\nu)/2))\Gamma((s+\nu)/2)}{\Gamma((s+\rho_0)/2)\Gamma((s-\rho_0+\delta)/2)} \tag{7.2}$$

holds.

Proof. (1) Let $\mathfrak{S}^T$ be the Siegel set introduced in the proof of Proposition 1. We fix $T > 0$ such that $\mathsf{G}(\mathbb{A}) = \mathsf{G}(\mathbb{Q})\mathfrak{S}^T$. Then as is proved in the course of the proof of Proposition 2, there exists a constant $C_{B_2} > 0$ such that $|F_s(g)| \leqslant C_{B_2}|\mathrm{N}(t(g))|_{\mathbb{A}}^{\delta/2}$ ($\forall g \in \mathfrak{S}^T$, $\forall s \in B_2$). By examining the constant term of $E(\nu)$, we also have a constant $C_{B_1} > 0$ such that $|E(\nu : g)| \leqslant C_{B_1}|\mathrm{N}(t(g))|_{\mathbb{A}}^{(|\mathrm{Re}(\nu)|+\rho_0)/2}$ ($\forall g \in \mathfrak{S}^T$, $\forall \nu \in B_1$). Thus the integral $\int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} |F_s(g)E(\nu : g)|d\dot{g}$ is dominated by $C_{B_1}C_{B_2}\mathrm{mes}(\omega)\int_T^\infty t^{\delta+|\mathrm{Re}(\nu)|-\rho_0}\frac{dt}{t}$, which is finite if $|\mathrm{Re}(\nu)| < \rho_0 - \delta$.

(2) We have

$$\begin{aligned}\langle F|E\rangle(s,\nu) &= \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} \sum_{\gamma\in\mathsf{H}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})} \phi_s(\gamma g)E(\nu : \gamma g)\,d\dot{g} \\ &= \int_{\mathsf{H}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} \phi_s(g)E(\nu : g)\,d\dot{g}\end{aligned} \tag{7.3}$$

$$= \int_{\mathsf{H}(\mathbb{A})\backslash\mathsf{G}(\mathbb{A})} \phi_s(g)\mathcal{P}(\nu\,;g)\,d\mu_{X_{\mathbb{A}}}(\dot g)$$
$$= \int_0^\infty \left(\frac{1}{\cosh t}\right)^{s+\rho_0} \mathcal{P}(\nu\,;a_t)(\sinh t)^{\delta-1}(\cosh t)^{2\rho_0-\delta+1}dt$$

with $\mathcal{P}(\nu\,;g) := \mathcal{P}(\nu\,;\psi,\xi\,:\,g\,;1)$. By Proposition 1, the function $\mathcal{P}(\nu\,;g)$ in g_∞ is C^∞ and $R_\Omega\mathcal{P}(\nu\,;g) = (\nu^2-\rho_0^2)\mathcal{P}(\nu\,;g)$ since $R_\Omega E(\nu) = (\nu^2-\rho_0^2)E(\nu)$. Using the fact that $\mathcal{P}(\nu\,;g)$ is left $\mathsf{H}(\mathbb{R})$-invariant and right K_∞-invariant, combined with the formula (7.1), we have the ordinary differential equation for the function $\mathcal{P}(\nu\,;a_t)$ in t, which can be solved quite easily as

$$(7.4)\qquad \mathcal{P}(\nu\,;a_t) = C_1\cdot(\cosh t)^{-(\nu+\rho_0)}{}_2F_1\left(\frac{\nu+\rho_0}{2},\frac{\nu-\rho_0+\delta}{2}\,;\frac{\delta}{2}\,;\mathrm{th}^2 t\right),\quad t\in\mathbb{R}$$

with a constant C_1. By putting $t=0$, we have $C_1 = \int_{\mathsf{H}(\mathbb{Q})\backslash\mathsf{H}(\mathbb{A})} E(\nu\,:\,h)\,d\dot h$. By the substitution of the formula (7.4) to the final form of $\langle F|E\rangle(s,\nu)$ given by (7.3) and by the change of variables $z=\mathrm{th}^2 t$, we have

$$\langle F|E\rangle(s,\nu) = C_1\int_0^\infty (\cosh t)^{-(s+\nu+\delta-1)}(\sinh t)^{\delta-1}F_1\left(\frac{\nu+\rho_0}{2},\frac{\nu-\rho_0+\delta}{2}\,;\frac{\delta}{2}\,;\mathrm{th}^2 t\right)dt$$
$$= C_1\int_0^1 F_1\left(\frac{\nu+\rho_0}{2},\frac{\nu-\rho_0+\delta}{2}\,;\frac{\delta}{2}\,;z\right)z^{\delta/2-1}(1-z)^{(s+\nu)/2-1}dz\,.$$

By [7, 7.512,4 (p. 849)], we obtain the formula (7.2). □

7.2. Calculation of inner product, via constant terms and truncation

For $0\leqslant T<T'\leqslant+\infty$, we first introduce an auxiliary series by

$$(7.5)\quad e^{[T,T']}(\nu\,;g) := \sum_{\gamma\in\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})} \psi(t(\gamma g))\xi([\gamma g]_0)|\mathrm{N}(t(\gamma g))|^{(\nu+\rho_0)/2}\chi_{[T,T']}(t(\gamma g))\,,$$
$$g\in\mathsf{G}(\mathbb{A}),\ \nu\in\mathbb{C}\,.$$

If $T>0$, then by [1, Lemma 5.1], for given $g\in\mathsf{G}(\mathbb{A})$, the non-zero summand of $e^{[T,T']}(\nu\,;g)$ is finite in number. Hence the above series expression is valid for any ν. From (3.3) and (3.7), we have $E(\nu\,:\,g) = e^{[0,+\infty]}(\nu\,;g)$ for $\mathrm{Re}(\nu)>\rho_0$ and

$$(7.6)\qquad \bigwedge^{T} E(\nu\,:\,g) = E(\nu\,:\,g) - e^{[T,+\infty]}(\nu\,;g) - M(\nu)\,e^{[T,+\infty]}(-\nu\,;g),\quad \forall\nu\in\mathbb{C}\,.$$

We first prove a lemma.

LEMMA 16. *Given* $0\leqslant T<T'\leqslant+\infty$, *we put* $B^T:=\mathbb{C}$ *if* $T>0$ *and put* $B^0:=\{\nu\in\mathbb{C}\,|\,\mathrm{Re}(\nu)>\rho_0\}$. *Then for any continuous function* $f:\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})/K_{\mathbb{A}}\to\mathbb{C}$,

$$\int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} f(g)e^{[T,T']}(\nu\,:\,g)\,d\dot g$$
$$= \sum_{(i,j)\in C_E\times C_{S_0}} M_{\psi,\xi}(i,j)\int_T^{T'} f_{(\mathsf{P})}(\mathsf{m}(t_i r\,;m_{0,j}))r^{\nu-\rho_0}\frac{dr}{r},\quad \nu\in B^T$$

provided that the integrals are absolutely convergent. Here $f_{(\mathsf{P})}$ *denotes the constant term of* f *defined as* (5.1)*.*

Proof. As remarked above, for $\nu \in B^T$, the series expression (7.5) is valid. Hence by the integration formula (3.2), we have

$$\begin{aligned}(7.7)\qquad &\int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} f(g)e^{[T,T']}(\nu : g)\,d\dot{g}\\ &= \int_{\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} f(g)\,\xi([g]_0)\psi(t(g))|\mathrm{N}(t(g))|_{\mathbb{A}}^{(\nu+\rho_0)/2}\chi_{[T,T']}(t(g))\,d\dot{g}\\ &= \int_{\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1} \xi([m]_0)\,\psi(t(m))\,d^1\dot{m}\int_T^{T'} f_{(\mathsf{P})}(m\,\mathsf{m}(r\,;\,\mathtt{I}_q))r^{\nu-\rho_0}\frac{dr}{r}\\ &= \sum_{(i,j)\in C_E\times C_{S_0}} \mathrm{mes}(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{Q})\mathsf{m}(t_i\,;\,m_{0,j})K_{\mathbb{A}}^{\mathsf{M}})\,\xi(m_{0,j})\,\psi(t_i)\\ &\quad\times\int_T^{T'} f_{(\mathsf{P})}(\mathsf{m}(t_ir\,;\,m_{0,j}))r^{\nu-\rho_0}\frac{dr}{r}\,.\end{aligned}$$

Here to obtain the last equality we use the decomposition $\mathsf{M}(\mathbb{A})^1 = \bigsqcup_{(i,j)\in C_E\times C_{S_0}} \mathsf{M}(\mathbb{Q})\mathsf{m}(t_i\,;\,m_{0,j})K_{\mathbb{A}}^{\mathsf{M}}$ with $K_{\mathbb{A}}^{\mathsf{M}} := K_{\mathbb{A}}\cap\mathsf{M}(\mathbb{A})^1$. We fix a Haar measure of $E_{\mathbb{A}}^{(1)}$ such that $\mathrm{mes}(E^\times\backslash E_{\mathbb{A}}^{(1)}) = 1$. Then since the natural projection map $t_iU_{\mathbb{A}} \to E^\times\backslash E^\times t_iU_{\mathbb{A}}$ is $\#\mathscr{O}(t_i)^\times$ to 1 and since $\mathrm{mes}(t_iU_{\mathbb{A}}) = \mathrm{mes}(U_{\mathbb{A}}) = \mu(\mathscr{O})^{-1}$, we have $\mathrm{mes}(E^\times\backslash E^\times t_iU_{\mathbb{A}}) = \mu(\mathscr{O})^{-1}(\#\mathscr{O}(t_i)^\times)^{-1}$. Hence the volume factor in the right-hand side of the last equality of (7.7) becomes $\mu(\mathscr{O})^{-1}(\#(\mathscr{O}(t_i)^\times))^{-1}\ \mu(S_0\,;\,m_{0,j})$. This proves the formula. □

The L^2-inner product of two functions ξ and 1 on $\mathsf{G}_0(\mathbb{Q})\backslash\mathsf{G}_0(\mathbb{A})$ is denoted by $\langle\xi,1\rangle_{\mathsf{G}_0}$, i.e.,

$$(7.8)\qquad \langle\xi,1\rangle_{\mathsf{G}_0} := \sum_{j\in C_{S_0}} \mu(S_0\,;\,m_{0,j})\xi(m_{0,j})\,.$$

PROPOSITION 20. *Let* $T>0$.

(1) *The integral*

$$\left\langle\bigwedge^T F_s|E(\nu)\right\rangle := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})}\left(\bigwedge^T F_s(g)\right)E(\nu : g)\,d\dot{g}$$

converges absolutely and locally uniformly on $\{(s,\nu)\in\mathbb{C}^2\,|\,\mathrm{Re}(s)>\rho_0\}$.

(2) *Let* $(s,\nu)\in\mathbb{C}^2$ *be such that* $|\mathrm{Re}(\nu)|<\rho_0-\delta$, $\mathrm{Re}(s)>\rho_0$. *Then, we have*

(7.9)

$$\begin{aligned}\langle F|E\rangle(s,\nu) &= \left\langle\bigwedge^T F_s|E(\nu)\right\rangle + \langle\xi,1\rangle_{G_0}(\widehat{\Phi}^{[T,\infty]}(s,\nu,\psi)+M(\nu)\widehat{\Phi}^{[T,\infty]}(s,-\nu,\psi))\\ &\quad - M_{\psi,\xi}C(s)\left(\frac{T^{\nu-\rho_0+\delta}}{\nu-\rho_0+\delta}+M(\nu)\frac{T^{-\nu-\rho_0+\delta}}{-\nu-\rho_0+\delta}\right),\end{aligned}$$

where $M(\nu)$ is the meromorphic function introduced in the proof of Proposition 1. The function $(s,\nu)\mapsto \langle F|E\rangle(s,\nu)$ has a meromorphic continuation to $\mathrm{Re}(s) > |\mathrm{Re}(\nu)|$, $\mathrm{Re}(s) > \rho_0$.

Proof. (1) Fix a Siegel domain $\mathfrak{S}^T$ for $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$. Let $B \subset \{s \in \mathbb{C} |\, \mathrm{Re}(s) > \rho_0\}$ be an arbitrary compact set. From Proposition 2 and by [2, Lemma 1.4], for a given $N > 0$ there exists a constant $C_{B,N}$ such that $|\bigwedge^T F_s(g)| \leqslant C_{B,N}|\mathrm{N}(t(g))|_{\mathbb{A}}^{-N/2}$ ($\forall g \in \mathfrak{S}^T$, $\forall s \in B$). From this estimation, combined with the fact that $E(\nu : g)$ is of moderate growth on $\mathfrak{S}^T$, the assertion of (1) follows.

(2) From (7.6), $\langle F|E\rangle(s,\nu)$ equals

$$
\begin{aligned}
(7.10)\qquad &\int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} F_s(g) \bigwedge^T E(\nu : g)\, d\dot{g} + \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} F_s(g)\, e^{[T,+\infty]}(\nu\,;\, g)\, d\dot{g} \\
&+ M(\nu) \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} F_s(g)\, e^{[T,+\infty]}(-\nu\,;\, g)\, d\dot{g}\,.
\end{aligned}
$$

By [2, Corollary 1.2], the first term in the right-hand side of the equality is $\langle \bigwedge^T F_s | E(\nu)\rangle$. We can use Lemma 16 to have

$$
\begin{aligned}
(7.11)\qquad &\int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} F_s(g)\, e^{[T,+\infty]}(\pm\nu\,;\, g)\, d\dot{g} \\
&= \sum_{(i,j)\in C_E\times C_{S_0}} M_{\psi,\xi}(i,j) \int_T^\infty F_{s,(\mathsf{P})}(\mathsf{m}(t_i r\,;\, m_{0,j}))r^{\pm\nu-\rho_0}\frac{dr}{r} \\
&= \langle \xi, 1\rangle_{\mathsf{G}_0} \widehat{\Phi}^{[T,+\infty]}(s, \pm\nu, \psi) + \hat{\varphi}_{\psi,\xi}^{[T,+\infty]}(s, \pm\nu)\,.
\end{aligned}
$$

(Note $\Phi_s(\mathsf{m}(t\,;\, m_0))$ is independent of $m_0 \in \mathsf{G}_0(\mathbb{A})$ as remarked in the proof of Proposition 17.) These integrals are absolutely convergent by Proposition 18, Proposition 19 since $\mathrm{Re}(s) > \rho_0$, $|\mathrm{Re}(\nu)| < \rho_0 - \delta$. From (7.10), (7.11) and Proposition 19, (7.9) follows. By (1), the first term of the formula in the right-hand side of (7.9) is holomorphic on $\mathrm{Re}(s) > |\mathrm{Re}(\nu)|$, $\mathrm{Re}(s) > \rho_0$. Proposition 18 (1) ensures the holomorphicity of the third term and the fourth one on $\mathrm{Re}(s) > |\mathrm{Re}(\nu)|$, $\mathrm{Re}(s) > \rho_0$. Since the remaining terms are meromorphic there, we have the last assertion of (2). □

Let us examine the integral $\langle \bigwedge^T F_s | E(\nu)\rangle$ on the domain $\mathrm{Re}(s) > \rho_0$, $\mathrm{Re}(\nu) > \rho_0$, where we can use the series expression of $E(\nu)$. Before that, we introduce an auxiliary integral. For $T > 0$, $(s,\nu) \in \mathbb{C}^2$ with $\mathrm{Re}(s) > \rho_0$ and $g \in \mathsf{G}(\mathbb{A})$, we set

$$
\Lambda^T(s,\nu\,;\, g) := \int_0^\infty \int_{\mathsf{N}(\mathbb{A})} F_{s,(\mathsf{P})}(\mathsf{w}_0 u\, \mathsf{m}(r\,;\, \mathtt{I}_q)g)\chi_{[T,+\infty]}(t(\mathsf{w}_0 u\, \mathsf{m}(r\,;\, \mathtt{I}_q)g))r^{\nu-\rho_0}\frac{dr}{r}\, du\,,
$$

$$
\Lambda^T(s,\nu) := \sum_{(i,j)\in C_E\times C_{S_0}} M_{\psi,\xi}(i,j)\Lambda^T(s,\nu\,;\, \mathsf{m}(t_i\,;\, m_{0,j}))\,.
$$

LEMMA 17. *Let* $\mathrm{Re}(s) > \rho_0$. *There exists positive constant* $N_1 > \rho_0$ *and* R *such that for* $\mathrm{Re}(\nu) > N_1$

(1) *the double integral* $\Lambda^T(s, \nu\,; g)$ *is absolutely convergent for* $T > R$,

(2) *and we have*

$$\lim_{T\to+\infty} \Lambda^T(s, \nu) = 0\,.$$

Proof. Let B be a compact set of $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \rho_0\}$. If we take R sufficiently large, we can use the estimate in Proposition 17 to have

(7.12)

$$\begin{aligned}
&\int_0^\infty \int_{\mathsf{N}(\mathbb{A})} |F_{s,(\mathsf{P})}(\mathtt{w}_0\, u\, \mathsf{m}(r\,;\, \mathtt{I}_q)g)|\chi_{[T,\infty]}(t(\mathtt{w}_0\, u\, \mathsf{m}(r\,;\, \mathtt{I}_q)g))r^{\mathrm{Re}(\nu)-\rho_0}\frac{dr}{r}\, du \\
&\leqslant C_{R,B}\int_0^\infty \int_{\mathsf{N}(\mathbb{A})} |\mathrm{N}(t(\mathtt{w}_0\, u\, \mathsf{m}(r\,;\, \mathtt{I}_q)g)|_{\mathbb{A}}^{\delta/2}\chi_{[T,\infty]}(t(\mathtt{w}_0\, u\, \mathsf{m}(r\,;\, \mathtt{I}_q)g))r^{\mathrm{Re}(\nu)-\rho_0}\frac{dr}{r}\, du
\end{aligned}$$

for all $s \in B$ and all $T > R$ with some constant $C_{R,B} > 0$. Put $r_g = |\mathrm{N}(t(g))|_{\mathbb{A}}^{1/2}$. Using the Iwasawa decomposition $g = \mathsf{m}(t(g)\,; [g]_0)\, u_g\, \mathsf{k}(g)$, we make a change of variable by

$$v := \mathsf{m}(t(g)\, r\,; [g]_0)^{-1} u\, \mathsf{m}(t(g)\, r\,; [g]_0) u_g\,, \quad a := r_g \cdot r\,,$$

then the last integral in (7.12) becomes

$$\begin{aligned}
(7.13)\qquad & r_g^{\rho_0}\int_0^\infty \int_{\mathsf{N}(\mathbb{A})} |\mathrm{N}(t(\mathtt{w}_0\, \mathsf{m}(a\,;\, \mathtt{I}_q))v)|_{\mathbb{A}}^{\delta/2}\chi_{[T,\infty]}(t(\mathtt{w}_0\, \mathsf{m}(a\,;\, \mathtt{I}_q)v))a^{\mathrm{Re}(\nu)}\frac{da}{a}\, dv \\
&= r_g^{\rho_0}\int_0^\infty \int_{\mathsf{N}(\mathbb{A})} |\mathrm{N}(t(\mathtt{w}_0 u)|_{\mathbb{A}}^{\delta/2}\chi_{[Tr,\infty]}(t(\mathtt{w}_0 u))r^{\mathrm{Re}(\nu)-\delta/2}\frac{dr}{r}\, du\,.
\end{aligned}$$

Let $I(T)$ denote the integral in the right-hand side of (7.13). Then it suffices to prove that $I(T) < \infty$ and $\lim_{T\to\infty} I(T) = 0$. By [15, Lemma 2.7 IV (p. 147)], there exists a positive constant c_0 such that $c_0 \geqslant |\mathrm{N}(t(\mathtt{w}_0\, u))|_{\mathbb{A}}^{1/2}$ $(\forall u \in \mathsf{N}(\mathbb{A}))$. Hence we have

(7.14)

$$\begin{aligned}
I(T) &\leqslant c_0^\delta \int_0^{c_0T^{-1}} \left(\int_{\mathsf{N}(\mathbb{A})} \chi_{[Tr,\infty]}(t(\mathtt{w}_0\, u))\, du\right) r^{\mathrm{Re}(\nu)-\delta/2}\frac{dr}{r} \\
&= c_0^\delta T^{-\mathrm{Re}(\nu)+\delta/2}\int_0^{c_0} \left(\int_{\mathsf{N}(\mathbb{A})} \chi_{[r,\infty]}(t(\mathtt{w}_0\, u))\, du\right) r^{\mathrm{Re}(\nu)-\delta/2}\frac{dr}{r} \\
&= c_0^\delta T^{-\mathrm{Re}(\nu)+\delta/2}\int_0^{c_0} \int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})} \left(\sum_{\delta\in\mathsf{N}(\mathbb{Q})} \chi_{[r,\infty]}(t(\mathtt{w}_0\, \delta u))\right) du\, r^{\mathrm{Re}(\nu)-\delta/2}\frac{dr}{r}\,.
\end{aligned}$$

Here to obtain the first equality we made a change variable by $Tr \to r$. From [1, Lemma 5.1 (p. 936)], there exists a positive constant c_1 and a constant $N_0 \in \mathbb{R}$ such that

$$(7.15)\qquad \sum_{\delta\in\mathsf{N}(\mathbb{Q})} \chi_{[r,\infty]}(t(\mathtt{w}_0\delta n)) \leqslant c_1 r^{N_0}\,, \quad \forall n \in \mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})\,, \forall r \in (0, c_0)\,.$$

Therefore from (7.14) and (7.15), we have

$$I(T) \leqslant c_0^{\delta} c_1 T^{-\mathrm{Re}(\nu)+\delta/2} \int_0^{c_0} r^{\mathrm{Re}(\nu)-\delta/2+N_0} \frac{dr}{r}\,.$$

If $\mathrm{Re}(\nu) > \delta/2 - N_0$, this integral is finite, and consequently, we get the estimate

$$I(T) \leqslant c_2 T^{-\mathrm{Re}(\nu)+\delta/2}, \quad \forall T > R$$

with a constant c_2. Moreover if $-\mathrm{Re}(\nu) + \delta/2 < 0$, $\lim_{T\to\infty} I(T) = 0$. This completes the proof. □

From now on we fix a positive constant N_1 so that Lemma 17 is valid.

PROPOSITION 21. *Let* $\mathrm{Re}(s) > \mathrm{Re}(\nu) > N_1$. *Then we have*

$$\left\langle \bigwedge^{T} F_s | E(\nu) \right\rangle = \langle \xi, 1\rangle_{G_0} \widehat{\Phi}^{[0,T]}(s, \nu, \psi) + M_{\psi,\xi} C(s) \frac{T^{\nu-\rho_0+\delta}}{\nu - \rho_0 + \delta} - \Lambda^T(s, \nu) \tag{7.16}$$

Proof. Since $\mathrm{Re}(\nu) > \rho_0$, we can use the series expression of $E(\nu)$ and apply Lemma 16 to have

$$\left\langle \bigwedge^{T} F_s | E(\nu) \right\rangle = \sum_{(i,j)\in C_E\times C_{S_0}} M_{\psi,\xi}(i, j) \int_0^{\infty} \left(\bigwedge^{T} F_s\right)_{(\mathsf{P})} (\mathsf{m}(t_i r\,; m_{0,j})) r^{\nu-\rho_0} \frac{dr}{r} \tag{7.17}$$

By the Bruhat decomposition $\mathsf{G}(\mathbb{Q}) = \mathsf{P}(\mathbb{Q}) \sqcup \mathsf{P}(\mathbb{Q})\mathrm{w}_0\mathsf{N}(\mathbb{Q})$, we have

$$\begin{aligned}
&\left(\bigwedge^{T} F_s\right)_{(\mathsf{P})}(g) \\
&= F_{s,(\mathsf{P})}(g) - \int_{\mathsf{N}(\mathbb{Q})\backslash \mathsf{N}(\mathbb{A})} \left(\sum_{\gamma\in \mathsf{P}(\mathbb{Q})\backslash \mathsf{G}(\mathbb{Q})} F_{s,(\mathsf{P})}(\gamma u g)\chi_{[T,\infty]}(t(\gamma u g)) \right) d\dot{u} \\
&= F_{s,(\mathsf{P})}(g)(1 - \chi_{[T,\infty]}(t(g))) - \int_{\mathsf{N}(\mathbb{A})} F_{s,(\mathsf{P})}(\mathrm{w}_0 u g)\chi_{[T,\infty]}(t(\mathrm{w}_0 u g))\, du\,.
\end{aligned}$$

Hence putting this new expression of $(\bigwedge^T F_s)_{(\mathsf{P})}(g)$ back to (7.17) and using Lemma 17 (1), we have (7.16) by the obvious relation $\chi_{[0,T]}(t) = 1 - \chi_{[T,\infty]}(t)$. □

PROPOSITION 22. *Let* $\mathrm{Re}(s) > \mathrm{Re}(\nu) > N_1$. *Then we have*

$$\begin{aligned}
\langle F|E\rangle(s, \nu) = \langle \xi, 1\rangle_{G_0} (\widehat{\Phi}^{[0,\infty]}(s, \nu, \psi) + M(\nu)\widehat{\Phi}^{[T,\infty]}(s, -\nu, \psi)) \\
+ M(\nu)\, M_{\psi,\xi}\, C(s) \frac{T^{-\nu-\rho_0+\delta}}{-\nu - \rho_0 + \delta} - \Lambda^T(s, \nu)\,.
\end{aligned} \tag{7.18}$$

Proof. This follows immediately from Proposition 20 and Proposition 21. One should note that the fourth term in the right-hand side of (7.9), which diverges when $T \to \infty$, is canceled by the same term appearing in the right-hand side of (7.16). □

THEOREM 2. *For* $\mathrm{Re}(s) > \mathrm{Re}(\nu) > N_1$, *we have*

$$\langle F|E\rangle(s,\nu) = \langle \xi, 1\rangle_{\mathsf{G}_0} \widehat{\Phi}^{[0,+\infty]}(s,\nu,\psi).$$

Proof. This follows from Proposition 22 by letting $T \to \infty$. Indeed, since $\mathrm{Re}(\nu) > N_1$, and $\mathrm{Re}(s) > \rho_0$, by Proposition 18 (1) and Lemma 17 (2), all terms in the right-hand side of (7.18) except the first one approach 0 when T getting large. □

8. The main theorem

THEOREM 3. *Assume* $q > 2\delta^{-1}$. *Let* H *be the stabilizer in* G *of a vector* $\eta = \eta(\mathsf{a}) \in S^{-1}\mathscr{O}^{q+2}$. *Put* $\Delta = \Delta(\mathsf{a})$. *Suppose that all prime divisors of* Δ *are ramify or innert in* E *and that* $2 \notin R(E)$ *if* $\delta = 2$. *Let* $\psi \in \hat{C}_E$ *be a Grössencharacter and* $\xi : C_{S_0} \to \mathbb{C}$ *a function. Then the integral*

$$\mathcal{P}_{\mathsf{H}}(E(\nu\,;\psi,\xi)) := \int_{\mathsf{H}(\mathbb{Q})\backslash \mathsf{H}(\mathbb{A})} E(\nu\,;\psi,\xi : h)\, d\dot{h}$$

converges absolutely on $|\mathrm{Re}(\nu)| < \rho_0 - \delta$ *and given as follows.*

(1) *When* δq *is even, let* χ_{D_0} *be the Kronecker character of the quadratic field* $\mathbb{Q}(\sqrt{D_0})$ *with* $D_0 := (-1)^{\frac{\delta q}{2}} \det Q_0$. *Put*

$$C(S_0,\mathscr{O},\mathsf{a}) = \mu(\mathscr{O})^{-1}(\det Q_0)^{-1/2} 2^{(\rho_0+\delta)/2+\delta q-1} \pi^{1-\delta/2+\rho_0} \Delta^{\rho_0/2} D_{\mathscr{O}}^{(1-\delta q)/\delta-1/2}.$$

Then $\mathcal{P}_{\mathsf{H}}(E(\nu\,;\psi,\xi))$ *equals*

(8.1)

$$C(S_0,\mathscr{O},\mathsf{a})\,\langle\xi,1\rangle_{\mathsf{G}_0} (2^{-1/2}\Delta^{1/2} D_{\mathscr{O}}^{1/\delta})^{\nu}\,\psi(\mathfrak{d}_E)^{-1}\,\frac{\Gamma((\nu-\rho_0+\delta)/2)}{\Gamma((\nu+\rho_0-\delta+2)/2)}$$
$$\times \frac{L((\nu-\rho_0+\delta)/2\,;\boldsymbol{\lambda}_{\psi})}{L(\nu+1\,;\chi_{D_0})} \prod_{p|2\Delta\det(Q_0)} \Xi^{(p)}_{S_0,\Delta}(\nu-\rho_0+\delta) L^{(p)}(\nu+1\,;\chi_{D_0})(1-p^{-(\nu-\rho_0+\delta)}).$$

Here $L(\nu\,;\chi_{D_0})$ *denotes the Dirichlet* L*-function of* χ_{D_0}. *For each prime* p, $L^{(p)}(\nu\,;\chi_{D_0})$ *is the* p*-th local factor of* $L(\nu\,;\chi_{D_0})$.

(2) *When* $\delta = 1$ *and* q *is odd, let* χ_{D_0} *be the Kronecker character of the quadratic field* $\mathbb{Q}(\sqrt{D_0})$ *with* $D_0 := (-1)^{\frac{q-1}{2}}\Delta\,\det(Q_0)$. *Then* $\mathcal{P}_{\mathsf{H}}(E(\nu\,;1,1))$ *equals*

(8.2)

$$(\det Q_0)^{-1/2} 2^{(q-\rho_0+\nu)/2} \pi^{\rho_0+1/2} \Delta^{(\nu+\rho_0)/2} \frac{\Gamma((\nu-\rho_0+1)/2)}{\Gamma((\nu+\rho_0+1)/2)}$$
$$\times \frac{\zeta(\nu-\rho_0+1)L(\nu+1/2\,;\chi_{D_0})}{\zeta(2\nu+1)} \prod_{p|2\Delta\det(Q_0)} \frac{\Xi^{(p)}_{S_0,\Delta}(\nu-\rho_0+1)\zeta^{(p)}(2\nu+1)}{\zeta^{(p)}(\nu-\rho_0+1)L^{(p)}(\nu+1/2\,;\chi_{D_0})}.$$

Proof. First note that the function $\nu \mapsto \mathcal{P}_{\mathsf{H}}(E(\nu))$ on $|\mathrm{Re}(\nu)| < \rho_0 - \delta$ has an analytic continuation to the domain $\mathrm{Re}(s) > |\mathrm{Re}(\nu)|$, $\mathrm{Re}(s) > \rho_0$ by the formula (7.2) in Theorem 1 and Proposition 20 (2) with keeping the formula (7.2) valid. For (s,ν) satisfying

$\mathrm{Re}(s) > \mathrm{Re}(\nu) > N_1$, we have another expression of $\langle F|E\rangle(s,\nu)$ given by Theorem 2 and Proposition 18 (2) . First equating these two expressions and then dividing a suitable gamma factor, we obtain (8.1) or (8.2) for $\mathrm{Re}(s) > \mathrm{Re}(\nu) > N_1$. On the other hand, by Lemma 15 combined with the meromorphicity of zeta-functions $L(\nu\,;\boldsymbol{\lambda}_\psi)$ and $L(\nu\,;\chi_{D_0})$, the expression in the right-hand side of (8.1) or (8.2) is meromorphic in $\nu \in \mathbb{C}$. Hence the same formula should remain valid on $|\mathrm{Re}(\nu)| < \rho_0 - \delta$. □

REMARK.

When $q \leqq 2^{-1}\delta$, the integral $\mathcal{P}_{\mathsf{H}}(E(\nu;\,\psi;\,\xi))$ has no convergence-region at all. There are several ways to consider the regularization of the divergent integral $\mathcal{P}_{\mathsf{H}}(E(\nu;\,\psi;\,\xi))$ ([5], [10], [17]). We remark that after some extra work, by slightly extending the argument in section 7, we obtain the same formula in Theorem 3 for the regularization of $\mathcal{P}_{\mathsf{H}}(E(\nu;\,\psi;\,\xi))$ in the sense of [5].

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